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An adaptive convex combination of APA and ZA-APA for identifying systems having variable sparsity and correlated input

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ABSTRACT

In this paper, we present an efficient algorithm for identifying and tracking the impulse response of a sparse system that exhibits time varying sparseness and is driven by correlated input. The proposed method convexly combines the outputs of two filters, namely, the sparsity unaware affine projection algorithm (APA), and the sparsity aware zero attracting affine projection algorithm (ZA-APA), each trying to identify the same system using the same input. The combining parameter is adapted by following a steepest descent of the error variance at the convex combination output. A detailed performance analysis of the proposed combination is carried out, which reveals that while for highly non-sparse and highly sparse systems, the proposed combination converges respectively to the APA and ZA-APA (i.e., better of the two filters under the given levels of sparsity), for certain sparsity ranges, it leads to a combination filter that performs better than both the constituent filters. The claims made are validated by exhaustive simulation studies using white, colored as well as speech inputs.

1. Introduction

Last decade has seen a flurry of research activities in the area of sparse adaptive filters, motivated primarily by application areas like acoustic and network echo cancellation, underwater communication, high-definition television (HDTV) etc [1]. Unlike conventional, sparsity unaware adaptive filters like the least mean square (LMS), the recursive least squares (RLS) and their various variants [2], these filters deploy sparsity aware coefficient adaptation, and thereby achieve significant improvement in convergence performance, both in terms of convergence speed and steady state excess mean square error (EMSE) when the system is sparse. A prominent category in this context is the zero-attracting (ZA) family, in particular, the zero-attracting least mean square (ZA-LMS) [3] and the zero attracting normalized LMS (ZA-NLMS) [4] algorithms, where a ℓ_1-norm penalty of the coefficient vector is added to the LMS/NLMS (i.e., normalized LMS) cost function. Minimization of the cost function results in the introduction of a zero-attracting term in the weight update equation which pulls all the coefficients towards zero, thereby improving the convergence rate and steady state EMSE for a system that is highly sparse. However, as the zero-attraction is applied uniformly to all coefficients, if the system is less sparse, there will be zero attraction on the active taps (i.e., taps corresponding to the non-zero coefficients of the system impulse response) also, which will enhance the EMSE. To overcome this problem, a reweighted version of the ZA-LMS/NLMS, namely, reweighted ZA-LMS/NLMS (RZA-LMS/RZA-NLMS) has been proposed which tries to restrict the shrinkage mostly to the inactive (i.e., zero-valued) taps. Parameter selection, however, has been a tricky issue in the reweighted algorithms, especially when the system has several coefficients that have magnitudes neither zero nor very high. An alternative approach that is free of the above parameter selection issue was proposed in [5,6], which convexly combines the outputs of two adaptive filters, one sparsity aware and the other sparsity unaware, each trying to identify the same system. The combining parameter of the convex combination is adapted such that for highly sparse systems, major share in the output comes from the sparsity aware filter, while for systems that are non-sparse or less sparse, the same comes from the sparsity unaware filter. Initially, the combination was restricted to LMS and ZA-LMS [5], and later was extended to NLMS and ZA-NLMS [6].

For applications involving correlated input, e.g., acoustic echo cancellation, convergence of the NLMS/LMS algorithms, however, slows down as the correlation in the input increases. A more appropriate choice in this case is the well known affine projection algorithm (APA) [7–11] which has a mechanism to decorrelate the input with reasonable increase in computational cost [12]. In re-

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cent past, the APA has been used in a convex combination framework in various non-sparse contexts [13–15]. Separately, for sparse system applications, the zero-attracting versions of APA, namely, the zero-attracting APA (ZA-APA) and the reweighted ZA-APA (RZA-APA) have been proposed in [16]. In [17], the mean square convergence behavior of the ZA-APA has been studied, and a selection criterion for the zero attractor has been derived. These are followed by two recent algorithms, namely the affine projection algorithm for sparse system identification (APA-SSI) and the quasi APA-SSI (QAPA-SSI) [18–20] which employ penalty functions that are approximations of the ℓ_0-norm of the parameter vector. In [21], a non-uniform sparsity promoting constraint has been used to modify the aforementioned adaptive algorithms by means of exact computation of the statistical expectation of the absolute values of each adaptive filter coefficient at each iteration. In [22], the ℓ_0-norm regularized APA algorithms have been modified for non-circular complex signals. As shown in [18–20], APA-SSI and QAPA-SSI outperform the ZA category of APA, especially for ideal sparse systems (i.e., filter taps taking the values from $\{0, 1\}$). However, for systems that have non-zero coefficients having close to zero magnitudes, these algorithms as well as the ones proposed in [21,22] suffer from parameter selection issues.

In this paper, we provide a more robust solution to identify a sparse system with time varying sparseness, by a convex combination of the sparsity unaware APA and the sparsity aware ZA-APA [16]). Analysis of such a combination is, however, extremely difficult due to the presence of normalized update terms in the update equation of both the filters and a nonlinear term in the update equation of the ZA-APA. To evaluate its behavior analytically, we use the NLMS with orthogonal correction factors (NLMS-OCF) model [8] for the APA and an elegant scheme of angular discretization of continuous valued random vectors [24] that helps in reducing the aforesaid complexities. The analysis shows that while for systems that are highly sparse or highly non-sparse, the proposed combination converges to the ZA-APA or the APA based filter respectively (i.e., better of the two under the given sparsity conditions), for systems that lie between moderately sparse to moderately non-sparse, the proposed combination can, however, perform better than both the constituent filters. The same is also verified via extensive simulation studies under different sparsity conditions. Lastly, a similar combination of ZA-APA and APA has also been reported in [23], though its analysis as provided in [23] fails to predict the behavior of the combination for different ranges of sparsity. Also, the simulation results given in [23] also do not reveal existence of the “better than the best” case.

The rest of the paper is organized as follows. We describe the proposed algorithm in Section 2 and carry out a detailed performance analysis for mean and mean square convergence of the proposed algorithm in Section 3. In Section 4, experimental results are provided, and finally we close the paper with our main conclusions in Section 5.

2. Proposed algorithm

We consider here the problem of identifying a system that takes a zero mean input signal $u(n)$ and yields the observable output $d(n) = \mathbf{u}^T(n)\mathbf{w}_{opt} + \vartheta(n)$, where $\mathbf{u}(n) = [u(n), u(n-1), \cdots, u(n-L+1)]^T$ is the input data vector at time n, $\mathbf{w}_{opt}$ is a $L \times 1$ system impulse response vector (to be identified) which is known *a priori* to be sparse with variable sparsity and $\vartheta(n)$ is an observation noise with zero mean and variance σ_ϑ^2 which is taken to be i.i.d. and independent of input $u(m)$ for all n and m. As the sparsity of the system varies over a wide range, from being highly sparse to fully non-sparse, neither a sparsity aware algorithm (e.g. ZA-APA) nor a sparsity unaware algorithm (e.g. APA) alone will be enough to identify and then track the system. In order to address this problem, we take recourse to the philosophy of convex combination of two adaptive filters [25], proposed first in [26,27] and subsequently used in [5] and [6] for sparse system identification. In the proposed scheme, we convexly combine the outputs of two adaptive filters, Filter1 and Filter2, of which Filter1 updates a weight vector $\mathbf{w}_1(n)$ using the APA [7], with the update equation given as

$$\mathbf{w}_1(n+1) = \mathbf{w}_1(n) + \mu\mathbf{U}(n)\big(\epsilon\mathbf{I}_P + \mathbf{U}^T(n)\mathbf{U}(n)\big)^{-1}\mathbf{e}_1(n), \tag{1a}$$

and, Filter2 updates a tap-weight vector $\mathbf{w}_2(n)$ following the ZA-APA [16], with update equation given as

$$\begin{aligned}\mathbf{w}_2(n+1) =& \mathbf{w}_2(n) + \mu\mathbf{U}(n)\big(\epsilon\mathbf{I}_P + \mathbf{U}^T(n)\mathbf{U}(n)\big)^{-1}\mathbf{e}_2(n) \\ &- \rho sgn\big(\mathbf{w}_2(n)\big),\end{aligned} \tag{1b}$$

where μ is the adaptation step size (common for both the filters), ρ is the shrinkage parameter (usually a very small constant), $\mathbf{U}(n) = \big[\mathbf{u}(n), \mathbf{u}(n-1), \cdots, \mathbf{u}(n-P+1)\big]$ is the data matrix at index n, P is the projection order, $\mathbf{y}_l(n) \equiv [y_l(n), y_l(n-1), \cdots, y_l(n-P+1)]^T = \mathbf{U}^T(n)\mathbf{w}_l(n)$, $l = 1, 2$, where $y_l(n-r)$, $r = 0, 1, \cdots, P-1$, is the l^{th} filter output at time index $(n-r)$ keeping the filter coefficient vector as $\mathbf{w}_l(n)$, $\mathbf{d}(n) = [d(n), d(n-1), \cdots, d(n-P+1)]^T$ where $d(n)$ is the desired response, and $\mathbf{e}_l(n) \equiv [e_l(n), e_l(n-1), \cdots, e_l(n-P+1)]^T = \mathbf{d}(n) - \mathbf{y}_l(n)$, $l = 1, 2$.

In a convex combination, the final output $y(n)$ is formed from the two individual filter outputs $y_1(n)$ and $y_2(n)$ as $y(n) = \tau(n)y_1(n) + \big(1-\tau(n)\big)y_2(n)$, where $\tau(n) \in [0, 1]$ is a mixing parameter. In an adaptive convex combination, $\tau(n)$ is updated in time by following a steepest descent search on $e^2(n)$, where $e(n) = d(n) - y(n)$ is the error at the output of the combination. However, such adaptation does not guarantee $\tau(n)$ to lie between 0 and 1, and to circumvent this problem, $\tau(n)$ is expressed monotonically in terms of another variable $a(n)$ as $\tau(n) = \frac{1}{1+exp(-a(n))}$ (i.e., sigmoidal function). The variable $a(n)$ is then updated by following a steepest descent search on $e^2(n)$ and the corresponding update equation is given by [25],

$$a(n+1) = a(n) + \mu_a e(n)\big(y_1(n) - y_2(n)\big)\tau(n)\big(1-\tau(n)\big). \tag{2}$$

In practice, updation of $a(n)$ is restricted to a range $[-a^+, +a^+]$ (a^+: a large finite number) that limits the permissible range of $\tau(n)$ to $[1-\tau^+, \tau^+]$, where $\tau^+ = \frac{1}{1+exp(-a^+)}$.

3. Performance analysis of the proposed combination

As in [26], we introduce certain definitions that are useful in the analysis. For $l = 1, 2$, we thus define:

a) *Weight Error Vectors*: $\widetilde{\mathbf{w}}_l(n) = \mathbf{w}_{opt} - \mathbf{w}_l(n)$;
b) *Equivalent Weight Vector* for the combined filter: $\mathbf{w}(n) = \tau(n)\mathbf{w}_1(n) + \big(1-\tau(n)\big)\mathbf{w}_2(n)$;
c) *Equivalent Weight Error Vector* for the combined filter: $\widetilde{\mathbf{w}}(n) = \mathbf{w}_{opt} - \mathbf{w}(n) = \tau(n)\widetilde{\mathbf{w}}_1(n) + \big(1-\tau(n)\big)\widetilde{\mathbf{w}}_2(n)$;
d) *A Priori Errors*: $e_{a,l}(n) = \mathbf{u}^T(n)\widetilde{\mathbf{w}}_l(n)$ and $e_a(n) = \mathbf{u}^T(n)\widetilde{\mathbf{w}}(n)$. Clearly, $e_a(n) = \tau(n)e_{a,1}(n) + \big(1-\tau(n)\big)e_{a,2}(n)$ and $e(n) = e_a(n) + \vartheta(n)$;
e) *EMSE*: $J_{ex,l}(n) = E[e_{a,l}^2(n)]$, for $l = 1, 2$ and $J_{ex}(n) = E[e_a^2(n)]$;
f) *Cross EMSE*: $J_{ex,12}(n) = E[e_{a,1}(n)e_{a,2}(n)] \leq \sqrt{J_{ex,1}(n)}\sqrt{J_{ex,2}(n)}$ (from Cauchy–Schwartz inequality). This means $J_{ex,12}(n)$ cannot be greater than both $J_{ex,1}(n)$ and $J_{ex,2}(n)$ simultaneously,
g) *EMSE of the combination*: $J_{ex}(n) = E[e_a^2(n)] = E[\big(\tau(n)e_{a,1}(n) + \big(1-\tau(n)\big)e_{a,2}(n)\big)^2] = E\big[\tau^2(n)e_{a,1}^2(n) + (1-\tau(n))^2e_{a,2}^2(n) + 2\tau(n)(1-\tau(n))e_{a,1}(n)e_{a,2}(n)\big]$.

Lastly, the steady state values of $J_{ex}(n)$, $J_{ex,l}(n)$, $l=1,2$ and of $J_{ex,12}(n)$ are denoted by $J_{ex}(\infty)$, $J_{ex,l}(\infty)$ and $J_{ex,12}(\infty)$ respectively (theoretically, $J_{ex}(\infty)=\lim_{n\to\infty} J_{ex}(n)$, $J_{ex,l}(\infty)=\lim_{n\to\infty} J_{ex,l}(n)$ and $J_{ex,12}(\infty)=\lim_{n\to\infty} J_{ex,12}(n)$).

Taking expectation on both sides of (2), one can write

$$E[a(n+1)]=E[a(n)]+\mu_a E\big[e(n)\big(y_1(n)-y_2(n)\big)\\ \times\tau(n)\big(1-\tau(n)\big)\big]. \tag{3}$$

In order to analyze the dynamics of $E[a(n)]$, we assume the time index n to belong to the steady state of both Filter1 and Filter2 (i.e., n is sufficiently large). One can then assume that both $J_{ex,1}(n)$ and $J_{ex,2}(n)$ have converged to their steady state values $J_{ex,1}(\infty)$ and $J_{ex,2}(\infty)$ respectively, and in the same way, $J_{ex,12}(n)$ has converged to $J_{ex,12}(\infty)$ (conditions for convergence will be developed as part of our analysis). Then, replacing $e(n)$ by $e_a(n)+\vartheta(n)$ in (3) where $e_a(n)$ is defined above, noting that $y_1(n)-y_2(n)=e_{a,2}(n)-e_{a,1}(n)$, $\vartheta(n)$ is zero mean, i.i.d., and assuming, like [26], that in the steady state, $\tau(n)$ is independent of the *a priori errors* $e_{a,l}(n)$, it is easy to verify that for large n (theoretically as $n\to\infty$) [26],

$$E[a(n+1)]=E[a(n)]-\mu_a E\big[\tau^2(n)\big(1-\tau(n)\big)\big]\Delta J_1\\ +\mu_a E\big[\tau(n)\big(1-\tau(n)\big)^2\big]\Delta J_2, \tag{4}$$

where $\Delta J_1 = J_{ex,1}(\infty) - J_{ex,12}(\infty)$ and $\Delta J_2 = J_{ex,2}(\infty) - J_{ex,12}(\infty)$.

From Eq. (4), it is clear that dynamics of the evolution of $E[a(n)]$ with time depends on ΔJ_1 and ΔJ_2, or, equivalently, on $J_{ex,1}(\infty)$, $J_{ex,2}(\infty)$ and $J_{ex,12}(\infty)$.

Similarly, assuming that $\tau(n)$ is independent of the *a priori errors* $e_{a,l}(n)$ in the steady state, it is easy to verify that for large n (theoretically as $n\to\infty$) [26],

$$J_{ex}(n)=E[\tau^2(n)]J_{ex,1}(\infty)+E[(1-\tau(n))^2]J_{ex,2}(\infty)\\ +2E[\tau(n)(1-\tau(n))]J_{ex,12}(\infty). \tag{5}$$

From Eq. (5), it is clear that $J_{ex}(\infty)$ depends on $J_{ex,1}(\infty)$, $J_{ex,2}(\infty)$ and $J_{ex,12}(\infty)$.

Of these three, we evaluate $J_{ex,2}(\infty)$ and $J_{ex,12}(\infty)$ here, while $J_{ex,1}(\infty)$ has been worked out in [8] (also can be obtained by setting $\rho=0$ in the expression for $J_{ex,2}(\infty)$ we derive below). Evaluation of $J_{ex,2}(\infty)$ and $J_{ex,12}(\infty)$, however, is a very difficult task due to the presence of the normalizing term $\big(\epsilon\mathbf{I}_P+\mathbf{U}^T(n)\mathbf{U}(n)\big)^{-1}$ in the update equation of both Filter1 and Filter2, and the nonlinear term $\rho sgn\big(\mathbf{w}_2(n)\big)$ in the update equation of Filter2. In order to circumvent this difficulty, we adopt the following:

A. The NLMS-OCF model for the APA [8], [9],
B. A scheme of angular discretization of continuous valued random vector [24].

Of these, the NLMS-OCF model has been discussed in [8] and [9]. As per this, the update equations (1a) and (1b) can be expressed equivalently as per the following:

For Filter1,

$$\mathbf{w}_1(n+1)=\mathbf{w}_1(n)+\mu_{1,0}\mathbf{u}^0(n)+\mu_{1,1}\mathbf{u}^1(n)+\cdots\\ +\mu_{1,P-1}\mathbf{u}^{P-1}(n), \tag{6a}$$

and for Filter2,

$$\mathbf{w}_2(n+1)=\mathbf{w}_2(n)+\mu_{2,0}\mathbf{u}^0(n)+\mu_{2,1}\mathbf{u}^1(n)+\cdots\\ +\mu_{2,P-1}\mathbf{u}^{P-1}(n)-\rho sgn\big(\mathbf{w}_2(n)\big), \tag{6b}$$

where, $\big\{\mathbf{u}^i(n)|i=0,1,\cdots,P-1\big\}$ is a set of orthogonal vectors obtained by Gram–Schmidt orthogonalization of the set $\big\{\mathbf{u}(n-j)|j=0,1,\cdots,P-1\big\}$, with $\mathbf{u}^0(n)\equiv\mathbf{u}(n)$ and for $k=1,2,\cdots,P-1$, $\mathbf{u}^k(n)$ is the projection error vector associated with the orthogonal projection of $\mathbf{u}(n-k)$ on the subspace spanned by $\big\{\mathbf{u}(n),\mathbf{u}(n-1),\cdots,\mathbf{u}(n-k+1)\big\}$. The coefficients $\mu_{l,k}$, $l=1,2$ are defined as follows:

$$\mu_{l,k}=\begin{cases}\frac{\mu e_l(n)}{\mathbf{u}^T(n)\mathbf{u}(n)} & \text{for } k=0 \text{ if } \mathbf{u}(n)\neq 0\ (=0 \text{ otherwise}),\\ \frac{\mu e_l^k(n)}{(\mathbf{u}^k(n))^T\,\mathbf{u}^k(n)} & \text{for } k=1,2,\cdots,P-1\\ & \text{if } \mathbf{u}^k(n)\neq 0\ (=0 \text{ otherwise}),\end{cases} \tag{7}$$

where

$$e_l(n)=d(n)-\mathbf{w}_l^T(n)\mathbf{u}(n),\\ e_l^k(n)=d(n-k)-\big(\mathbf{w}_l^k(n)\big)^T\mathbf{u}(n-k)\quad \text{for } k=1,2,\cdots,P-1,\\ \mathbf{w}_l^k(n)=\mathbf{w}_l(n)+\mu_{l,0}\mathbf{u}(n)+\mu_{l,1}\mathbf{u}^1(n)+\cdots+\mu_{l,k-1}\mathbf{u}^{k-1}(n). \tag{8}$$

The NLMS-OCF algorithm can be seen as the process of computing weight updates, using NLMS, based on the current input data vector $\mathbf{u}(n)$, as well as the orthogonal components from each of the previous $P-1$ input data vectors.

Angular Discretization of a Continuous Valued Random Vector [24]: Let $\mathbf{u}\in\mathbb{R}^L$ be a random vector with mean zero, i.e., $E(\mathbf{u})=\mathbf{0}$ and correlation matrix $E(\mathbf{u}\mathbf{u}^T)=\mathbf{R}$. Further, let $\mathbf{v}_i$, $i=0,1,\cdots,L-1$ be the orthonormal set of eigenvectors of $\mathbf{R}$ corresponding to the eigenvalues λ_i, $i=0,1,\cdots,L-1$. Defining $\mathbf{V}=[\mathbf{v}_0\,\mathbf{v}_1\,\cdots\,\mathbf{v}_{L-1}]$ and $\mathbf{D}=diag\{\lambda_i|i=0,1,\cdots,L-1\}$, one can then express $\mathbf{R}$ as $\mathbf{R}=\mathbf{V}\mathbf{D}\mathbf{V}^T=\sum_{i=0}^{L-1}\lambda_i\mathbf{v}_i\mathbf{v}_i^T$. In the angular discretization based approximation, the direction of $\mathbf{u}$ is discretized while the magnitude is kept unchanged. In particular, it is assumed that $\mathbf{u}$ can assume only one of the $2L$ directions, given by $\pm\mathbf{v}_i$, $i=0,1,\cdots,L-1$. Mathematically, $\mathbf{u}$ is then expressed as

$$\mathbf{u}=sr\mathbf{v}, \tag{9}$$

where $\mathbf{v}\in\{\mathbf{v}_i|i=0,1,\cdots,L-1\}$, with probability of $\mathbf{v}=\mathbf{v}_i$ given by p_i, $r=||\mathbf{u}||$, i.e., r has same distribution as that of $||\mathbf{u}||$ and $s\in\{1,-1\}$, with probability of $s=\pm1$ given by $Pr(s=\pm1)=\frac{1}{2}$. Further, the three elements s, r and $\mathbf{v}$ are assumed to be mutually independent. Note that as s is zero mean, $E[sr\mathbf{v}]=0$ and thus $E[\mathbf{u}]=0$ is satisfied trivially. To satisfy $E[\mathbf{u}\mathbf{u}^T]=\mathbf{R}$, the discrete probability p_i is taken as $p_i=\frac{\lambda_i}{Trace(\mathbf{R})}$, which satisfies $p_i\geq 0$, $\sum_{i=0}^{L-1}p_i=1$ and leads to $E[\mathbf{u}\mathbf{u}^T]=E[s^2r^2\mathbf{v}\mathbf{v}^T]=E[r^2]E[\mathbf{v}\mathbf{v}^T]=Trace(\mathbf{R})\sum_{i=0}^{L-1}p_i\mathbf{v}_i\mathbf{v}_i^T=\sum_{i=0}^{L-1}\lambda_i\mathbf{v}_i\mathbf{v}_i^T=\mathbf{R}$.

Lastly, for $\mathbf{u}$ consisting of uncorrelated entries with constant variance, say, σ_u^2, we have $\mathbf{R}=\sigma_u^2\mathbf{I}_L$, and the eigenvectors are given by the trivial basis $\{\mathbf{e}_i\,|\,i=0,1,\cdots,L-1\}$, where $\mathbf{e}_i$ is an $L\times1$ vector, with 1 in the i-th place and zero in all other places.

NLMS-OCF with angular discretization: As mentioned in [8] and [9], using the above angular discretization model, the weight update in (6)–(8) can be simplified. For this, first note that any data vector, say, $\mathbf{u}(n-k)$ can assume only one of the $2L$ directions $\pm\mathbf{v}_i$, $i=0,1,\cdots,L-1$, and also that the eigenvectors are mutually orthonormal. This means, in general, the data vectors $\mathbf{u}(n-k)$, $k=0,1,\cdots,P-1$ are mutually orthogonal (i.e., assuming that the data vectors are aligned to P different eigenvectors) and then, $\mathbf{u}^k(n)=\mathbf{u}(n-k)$, i.e., it is not required to carry out Gram–Schmidt orthogonalization of the set $\big\{\mathbf{u}(n-j)|j=0,1,\cdots,P-1\big\}$

and evaluate $\mathbf{u}^k(n)$. Also note that if $\mathbf{u}(n-k), k=1,2,\cdots,P-1$ is aligned with an eigenvector, say, $\mathbf{v}_j$ for some $j, 0 \le j \le L-1$ and at least one, or, more than one data vector belonging to the set $\{\mathbf{u}(n), \mathbf{u}(n-1), \cdots, \mathbf{u}(n-k+1)\}$ also is aligned to $\mathbf{v}_j$, then $\mathbf{u}^k(n) = \mathbf{0}$. For this, $\mu_{l,k} = 0$ as per (7).

In the light of the above, the weight update equations of Filter1 and Filter2 thus simplify to the following:

For Filter1,

$$\begin{aligned}\mathbf{w}_1(n+1) = \mathbf{w}_1(n) &+ \mu_{1,0}\mathbf{u}(n) + \mu_{1,1}\mathbf{u}(n-1) + \cdots \\ &+ \mu_{1,P-1}\mathbf{u}(n-P+1),\end{aligned} \tag{10}$$

and for Filter2,

$$\begin{aligned}\mathbf{w}_2(n+1) = \mathbf{w}_2(n) &+ \mu_{2,0}\mathbf{u}(n) + \mu_{2,1}\mathbf{u}(n-1) + \cdots \\ &+ \mu_{2,P-1}\mathbf{u}(n-P+1) - \rho sgn\big(\mathbf{w}_2(n)\big),\end{aligned} \tag{11}$$

where $\mu_{l,k}$ is given by (7). Note that due to the orthogonality of each of $\mathbf{u}(n), \mathbf{u}^1(n), \cdots, \mathbf{u}^{k-1}(n)$ with $\mathbf{u}(n-k)$ in this case, $e_l^k(n)$ in (8) will simplify to the following

$$\begin{aligned}&e_l^k(n) = d(n-k) - \mathbf{w}_l^T(n)\mathbf{u}(n-k) \\ &\quad \text{for } l = 1,2 \text{ and } k = 1,2,\cdots,P-1.\end{aligned} \tag{12}$$

3.1. *First order convergence analysis of the ZA-APA*

By substituting (7) in (11), we will have the update equation of ZA-APA as

$$\begin{aligned}\mathbf{w}_2(n+1) = \mathbf{w}_2(n) &+ \mu\frac{\mathbf{u}(n)e_2(n)}{\mathbf{u}^T(n)\mathbf{u}(n)} + \mu\frac{\mathbf{u}(n-1)e_2^1(n)}{\mathbf{u}^T(n-1)\mathbf{u}(n-1)} + \cdots \\ &+ \mu\frac{\mathbf{u}(n-P+1)e_2^{P-1}(n)}{\mathbf{u}^T(n-P+1)\mathbf{u}(n-P+1)} \\ &- \rho sgn\big(\mathbf{w}_2(n)\big),\end{aligned} \tag{13}$$

where $e_2^k(n)$, $k = 1,2,\cdot,P-1$ is given by (12). Denoting the weight deviation vector of the ZA-APA based adaptive filter at the n-th index as $\widetilde{\mathbf{w}}_2(n) = \mathbf{w}_{opt} - \mathbf{w}_2(n)$, from (12), we will have $e_2(n) = \mathbf{u}^T(n)\widetilde{\mathbf{w}}_2(n) + \vartheta(n)$ and $e_2^k(n) = \mathbf{u}^T(n-k)\widetilde{\mathbf{w}}_2(n) + \vartheta(n-k)$, $k = 1,2,\cdots,P-1$. Then, by substituting $e_2(n)$ and $e_2^k(n)$ $k = 1,2,\cdots,P-1$ in (13), the recursion for the weight deviation vector of the ZA-APA can then be obtained as follows:

$$\begin{aligned}\widetilde{\mathbf{w}}_2(n+1) = &\left(\mathbf{I}_L - \mu\sum_{j\in J_n}\frac{\mathbf{u}(n-j)\mathbf{u}^T(n-j)}{\mathbf{u}^T(n-j)\mathbf{u}(n-j)}\right)\widetilde{\mathbf{w}}_2(n) \\ &- \mu\sum_{j\in J_n}\frac{\mathbf{u}(n-j)\vartheta(n-j)}{\mathbf{u}^T(n-j)\mathbf{u}(n-j)} + \rho sgn\big(\mathbf{w}_2(n)\big),\end{aligned} \tag{14}$$

where $J_n \subseteq \{0,1,\cdots,P-1\}$ is the set of P or fewer indices j for which the input regressor vectors $\mathbf{u}(n-j)$ are orthogonal to each other. The orthogonalization process determines the indices forming the set J_n (note that J_n may change from experiment to experiment and is thus random). The equation (14) forms the basis for the performance analysis of the ZA-APA, where, using the aforesaid angular discretization model, we represent $\mathbf{u}(n)$ as

$$\mathbf{u}(n) = s(n)r(n)\mathbf{v}(n), \tag{15}$$

where $s(n)$, $r(n)$ and $\mathbf{v}(n)$ are equivalent to the variables s, r and $\mathbf{v}$ in (9) respectively. The first order convergence condition of the ZA-APA adaptive filter can then be worked out and is given in the theorem below:

Theorem 1. *For a zero-mean, white input signal $u(n)$ having covariance matrix $\mathbf{R} = \sigma_u^2\mathbf{I}_L$, a sufficient condition for the ZA-APA to converge in mean is $0 < \mu < 2$, and under this condition, the steady state mean weight for the i-th tap $\overline{w}_{2,i}(\infty)$ ($\equiv \lim_{n\to\infty} E[w_{2,i}(n)]$) is given as follows:*

1. *For the i-th active tap (i.e., $w_{opt,i} \neq 0$):*

$$\overline{w}_{2,i}(\infty) = w_{opt,i} - \frac{\rho}{\mu\beta}sgn\big(w_{opt,i}\big), \tag{16}$$

where $\beta = 1 - (1 - \frac{1}{L})^P$.

2. *For the i-th inactive tap (i.e., $w_{opt,i} = 0$):*

$$\overline{w}_{2,i}(\infty) = -\frac{\rho}{\mu\beta}E\big[sgn\big(w_{2,i}(n)\big)\big]\Big|_{\infty}, \tag{17}$$

where $E\big[sgn\big(w_{2,i}(n)\big)\big]\Big|_{\infty} = \lim_{n\to\infty} E\big[sgn\big(w_{2,i}(n)\big)\big]$.

Proof. Proof is provided in the Appendix A. □

From Theorem 1, for $i \in NZ$, we have

$$\overline{w}_{2,i}(\infty) = \begin{cases} w_{opt,i} - \frac{\rho}{\mu\beta}, & \text{if } sgn(w_{opt,i}) > 0 \\ w_{opt,i} + \frac{\rho}{\mu\beta}, & \text{if } sgn(w_{opt,i}) < 0, \end{cases} \tag{18}$$

and for $i \in Z$, we have $\overline{w}_{2,i}(\infty) = -\frac{\rho}{\mu\beta}E\big[sgn\big(w_{2,i}(n)\big)\big]\Big|_{\infty}$. That means $\overline{w}_{2,i}(\infty)$ is a biased estimate of $w_{opt,i}$ and the bias is proportional to shrinkage parameter ρ.

3.2. *Second order convergence analysis of the ZA-APA*

Next we focus on the second order convergence analysis of the ZA-APA. In particular, we aim to derive the closed form expression for the mean square deviation (MSD) of individual taps and then investigate the contribution of the active and inactive taps in the total EMSE.

From the definition (e),

$$\begin{aligned}J_{ex,2}(n) = E[e_{a,2}^2(n)] &= E[\mathbf{u}^T(n)\widetilde{\mathbf{w}}_2(n)\widetilde{\mathbf{w}}_2^T(n)\mathbf{u}(n)] \\ &= E[Trace\big(\mathbf{u}^T(n)\widetilde{\mathbf{w}}_2(n)\widetilde{\mathbf{w}}_2^T(n)\mathbf{u}(n)\big)] \\ &= Trace\big(E[\widetilde{\mathbf{w}}_2(n)\widetilde{\mathbf{w}}_2^T(n)\mathbf{u}(n)\mathbf{u}^T(n)]\big).\end{aligned}$$

Using the statistical independence of $\mathbf{u}(n)$ with $\widetilde{\mathbf{w}}_2(n)$, and substituting $\mathbf{R}$ by $\mathbf{VDV}^T$, we have $J_{ex,2}(n) = Trace\Big(\mathbf{K}_2(n)\mathbf{VDV}^T\Big) = \sum_{i=0}^{L-1}\lambda_i\widetilde{\lambda}_{2,i}(n)$, where $\widetilde{\lambda}_{2,i}(n) = [\mathbf{\Lambda}_2(n)]_{i,i}$ with $\mathbf{\Lambda}_2(n) = \mathbf{V}^T\mathbf{K}_2(n)\mathbf{V}$ and $\mathbf{K}_2(n) = E[\widetilde{\mathbf{w}}_2(n)\widetilde{\mathbf{w}}_2^T(n)]$. Note that $[\mathbf{K}_2(n)]_{i,i} = E[\widetilde{w}_{2,i}^2(n)]$: MSD of the i-th coefficient at index n, $i = 0,1,\cdots,L-1$. Also note that for white input, $\mathbf{V} = \mathbf{I}_L$ and thus $\mathbf{\Lambda}_2(n) = \mathbf{K}_2(n)$, implying, $\widetilde{\lambda}_{2,i}(n) \equiv$ MSD of the i-th coefficient.

For the proposed second order analysis, we generalize the aforesaid "independence assumption" to the following: we assume that $\mathbf{w}_2(n)$ is statistically independent of $d(n-j)$, $\mathbf{u}(n-j)$, $j = 0,1,\cdots,P-1$, since $\vartheta(n-j) = d(n-j) - \mathbf{w}_{opt}^T\mathbf{u}(n-j)$, the above implies that $\mathbf{w}_2(n)$ is also statistically independent of present and past noise samples $\vartheta(n-j)$, $j = 0,1,\cdots,P-1$. It is then possible to prove the following:

Theorem 2. *For a white input $u(n)$ having mean zero and variance $E[u^2(n)] = \sigma_u^2$, the ZA-APA exhibits stable EMSE performance under $0 < \mu < 2$, for which, the steady state EMSE of Filter2 (i.e., ZA-APA) is given as follows:*

$$J_{ex,2}(\infty) = J_{ex,1}(\infty) + J_{M,2}(\infty), \tag{19}$$

where

$$J_{ex,1}(\infty) = L\sigma_u^2\left(\frac{\mu}{2-\mu}\right)\sigma_\vartheta^2\frac{1}{\bar{r}^2},$$

$$J_{M,2}(\infty) = \frac{\sigma_u^2\rho^2}{\big(\mu(2-\mu)\beta\big)^2} \times \left(\begin{array}{l} -M\left(\frac{4}{\pi}(1-\mu\beta)^2\Omega + 2(2-\mu)(1-\mu\beta)\right) \\ +L(2-\mu\beta)(2-\mu) \end{array}\right), \tag{20}$$

$$\Omega = \sqrt{1+\left(\frac{\pi}{2}\frac{\mu(2-\mu)\beta}{(1-\mu\beta)^2}\right)\left(1+\frac{\mu^2\beta\sigma_\vartheta^2}{\bar{r}^2\rho^2}\right)} - 1,$$

$\bar{r}^2 = Trace(\mathbf{R})$, *and M is the number of inactive taps.*

Proof. Proof is given in the Appendix B. □

From (19) and (20), it is obvious that $J_{ex,1}(\infty) = J_{ex,2}(\infty)\big|_{\rho=0}$. In other words, $J_{ex,1}(\infty)$ is the steady state EMSE of Filter1, since, for $\rho = 0$, the ZA-APA reduces to the simple APA. This also conforms to the steady state EMSE expression for APA obtained in [8].

3.3. Cross excess mean square error analysis of ZA-APA and APA

From the definition (f),

$$\begin{aligned} J_{ex,12}(n) &= E[e_{a,1}(n)e_{a,2}(n)] = E[\mathbf{u}^T(n)\widetilde{\mathbf{w}}_1(n)\widetilde{\mathbf{w}}_2^T(n)\mathbf{u}(n)] \\ &= E\left[Trace\left(\mathbf{u}^T(n)\widetilde{\mathbf{w}}_1(n)\widetilde{\mathbf{w}}_2^T(n)\mathbf{u}(n)\right)\right] \\ &= Trace\left(E[\widetilde{\mathbf{w}}_1(n)\widetilde{\mathbf{w}}_2^T(n)\mathbf{u}(n)\mathbf{u}^T(n)]\right). \end{aligned}$$

Using the statistical independence of $\mathbf{u}(n)$ vis-a-vis $\widetilde{\mathbf{w}}_1(n)$ and $\widetilde{\mathbf{w}}_2(n)$, we have $J_{ex,12}(n) = Trace\left(\mathbf{K}_{12}(n)\mathbf{VDV}^T\right) = \sum_{i=0}^{L-1}\lambda_i\widetilde{\lambda}_{12,i}(n)$, where $\widetilde{\lambda}_{12,i}(n) = [\mathbf{\Lambda}_{12}(n)]_{i,i}$, with $\mathbf{\Lambda}_{12}(n) = \mathbf{V}^T\mathbf{K}_{12}(n)\mathbf{V}$ and $\mathbf{K}_{12}(n) = E[\widetilde{\mathbf{w}}_1(n)\widetilde{\mathbf{w}}_2^T(n)]$. Note that $[\mathbf{K}_{12}(n)]_{i,i} = E[\widetilde{w}_{1,i}(n)\widetilde{w}_{2,i}(n)]$: cross MSD of the i-th coefficient at index n, $i = 0, 1, \cdots, L-1$. Also note that for white input $\mathbf{V} = \mathbf{I}_L$ and thus $\mathbf{\Lambda}_{12}(n) = \mathbf{K}_{12}(n)$, implying, $\widetilde{\lambda}_{12,i}(n) \equiv$ cross MSD of the i-th coefficient.

Theorem 3. *For a white input $u(n)$ having mean zero and variance $E[u^2(n)] = \sigma_u^2$, the adaptive convex combination of APA and ZA-APA produces stable cross EMSE performance under $0 < \mu < 2$, for which, the steady state cross EMSE is given as follows:*

$$J_{ex,12}(\infty) = J_{ex,1}(\infty) + J_{M,12}(\infty), \tag{21}$$

where

$$J_{ex,1}(\infty) = L\sigma_u^2\left(\frac{\mu}{2-\mu}\right)\sigma_\vartheta^2\frac{1}{\bar{r}^2},$$

$$J_{M,12}(\infty) = -M\sigma_u^2\frac{\frac{\mu}{2-\mu}\sigma_\vartheta^2\frac{1}{\bar{r}^2}}{\Omega+1}. \tag{22}$$

Proof. Proof is provided in the Appendix C. □

3.4. Convergence analysis of $a(n)$ for various cases of sparsity level of the system

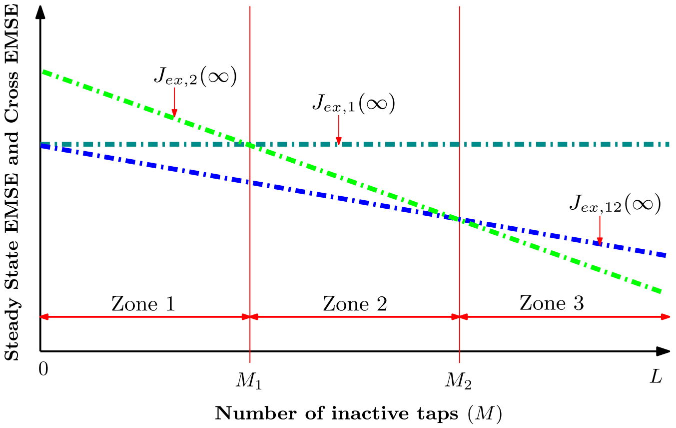


Fig. 1. Evolution of $J_{ex,1}(\infty)$, $J_{ex,2}(\infty)$, and $J_{ex,12}(\infty)$ against the sparsity level (i.e., number of inactive taps (M)) of the system.

From (20), we observe that $J_{M,2}(\infty)$ is a linear function of M (number of inactive taps) with a negative gradient, since $\left\{\frac{4}{\pi}(1-\mu\beta)^2\Omega + 2(2-\mu)(1-\mu\beta)\right\} > 0$. The following may then be inferred from this. First, for $M = 0$, i.e., for a fully non sparse system, from (20), $J_{M,2}(\infty) = \frac{L\rho^2\sigma_u^2(2-\mu\beta)}{\mu^2\beta^2(2-\mu)} > 0$. However, as M increases, $J_{M,2}(\infty)$ starts decreasing and reaches 0 at a point $M = M_1$, where

$$M_1 = \frac{L(2-\mu\beta)(2-\mu)}{c_1}, \tag{23}$$

with $c_1 = \frac{4}{\pi}(1-\mu\beta)^2\Omega + 2(2-\mu)(1-\mu\beta)$ (obtained by setting $J_{M,2}(\infty) = 0$). As M increases beyond M_1, $J_{M,2}(\infty)$ becomes negative and keeps decreasing linearly with M. Therefore, for $0 \le M < M_1$, we have $J_{ex,1}(\infty) < J_{ex,2}(\infty)$ and for $M_1 < M < L$, we have $J_{ex,1}(\infty) > J_{ex,2}(\infty)$. This is intuitive, as in ZA-APA, zero attraction uniformly shrinks all the taps. Thus, when the number of inactive taps M is high, ZA-APA is beneficial, whereas, when M is less, zero attraction on active coefficients results in enhancement of EMSE. The index M_1 can be considered as the boundary between a non-sparse system and sparse system.

On the other hand, from (22), it follows that for $M = 0$, $J_{M,12}(\infty) = 0$ and as M increases, $J_{M,12}(\infty)$ keeps decreasing linearly with M. This implies that for $0 < M \le L$, $J_{ex,1}(\infty) > J_{ex,12}(\infty)$. On the other hand, as $J_{M,12}(\infty)$ keeps decreasing linearly with M, it intersects the $J_{ex,2}(\infty)$ line at $M = M_2$, where

$$M_2 = \frac{L(2-\mu\beta)(2-\mu)}{c_2}, \tag{24}$$

with $c_2 = c_1 - \frac{\mu^3\beta^2(2-\mu)\sigma_\vartheta^2\frac{1}{\bar{r}^2}}{\rho^2(\Omega+1)}$ (obtained by solving $J_{ex,2}(\infty) = J_{ex,12}(\infty)$). Since, the second term in the RHS of c_2 is always positive, we have $c_2 < c_1$, or, equivalently $M_2 > M_1$. Note that as long as $\frac{(2-\mu\beta)(2-\mu)}{c_2} < 1$, we will have $M_2 < L$.

We plot $J_{ex,1}(\infty)$, $J_{ex,2}(\infty)$ and $J_{ex,12}(\infty)$ against M over the range $[0, L]$ in Fig. 1. Since the APA is sparsity unaware, its EMSE $J_{ex,1}(\infty)$ is constant for all values of M. The straight line representing the EMSE of ZA-APA, i.e., $J_{ex,2}(\infty)$ intersects the $J_{ex,1}(\infty)$ line at point $M = M_1$. On the other hand, $J_{ex,2}(\infty)$ and $J_{ex,12}(\infty)$ are seen to intersect at $M = M_2$. Fig. 1 shows the case where $M_2 < L$. In this case, the range $[0, L]$ can be divided into three zones, namely, (A). Zone 1 ($0 \le M < M_1$) (B). Zone 2 ($M_1 \le M < M_2$) (C). Zone 3 ($M_2 \le M \le L$). However, if $M_2 > L$, the two straight lines $J_{ex,2}(\infty)$ and $J_{ex,12}(\infty)$ do not intersect in the range of $[0, L]$. In that case, we will have only two zones, namely, Zone 1 and Zone 2 ($M_1 \le M \le L$). Convergence behavior of the proposed convex combination for the three zones above follows directly from [26]. In the following, we provide a detailed account of this.

3.4.1. Non-sparse systems

For an absolute non-sparse system, i.e., for smaller values of M, particularly for $M = 0$, we will have $J_{ex,1}(\infty) = J_{ex,12}(\infty)$ and

Table 1

$J_{ex}(\infty)$ and $\tau(\infty)$ as functions of M (i.e., number of inactive coefficients).

	$M = 0$	$0 < M < M_2$	$M_2 \leq M \leq L$
$M_2 \leq L$	$\tau(\infty) = \tau^+$ $J_{ex}(\infty) = J_{ex,1}(\infty)$	$\tau^+ \geq \tau(\infty) \geq 1 - \tau^+$ $J_{ex}(\infty) \leq \min\big(J_{ex,1}(\infty), J_{ex,2}(\infty)\big)$	$\tau(\infty) = 1 - \tau^+$ $J_{ex}(\infty) = J_{ex,2}(\infty)$
$M_2 > L$	$\tau(\infty) = \tau^+$ $J_{ex}(\infty) = J_{ex,1}(\infty)$	$\tau^+ \geq \tau(\infty) \geq 1 - \tau^+$ $J_{ex}(\infty) \leq \min\big(J_{ex,1}(\infty), J_{ex,2}(\infty)\big)$	

$J_{ex,2}(\infty) > J_{ex,12}(\infty)$. This implies $\triangle J_1 = 0$ and $\triangle J_2 > 0$. This is analogous to the case (1), section III of [26], for which Eq. (4) leads to

$$E[a(n+1)] = E[a(n)] + \mu_a E\big[\tau(n)\big(1-\tau(n)\big)^2\big]\triangle J_2. \tag{25}$$

Note that $\forall \tau(n) \in [0,1]$, the function $f_1\big(\tau(n)\big) = \tau(n)\big(1-\tau(n)\big)^2 \geq 0$. Then, for all $-a^+ \leq a(n) \leq a^+$ or equivalently $1-\tau^+ \leq \tau(n) \leq \tau^+$, we will have $f_1\big(\tau(n)\big) \geq f_1\big(\tau^+\big) = C$ and thus, $E[f_1\big(\tau(n)\big)] \geq C$, where $C = \tau^+\big(1-\tau^+\big)^2$. Substituting in (25), $E[a(n+1)] \geq E[a(n)] + \mu_a C \triangle J_2$. Since $\triangle J_2 > 0$, $E[a(n)]$ turns out to be an increasing sequence and $\lim_{n\to\infty} E[a(n)]$ converges to a^+, i.e., $\lim_{n\to\infty} a(n) = a^+$ almost surely, meaning $\lim_{n\to\infty} \tau(n) = \tau^+$ (almost surely) ≈ 1. Thus, for non-sparse systems, the proposed combination converges to the Filter1, i.e., APA in steady state.

3.4.2. Less sparse systems

In case of less sparse systems, i.e., for $0 < M < M_2$, we will have $J_{ex,1}(\infty) > J_{ex,12}(\infty)$ and $J_{ex,2}(\infty) > J_{ex,12}(\infty)$, implying $\triangle J_1 > 0$ and $\triangle J_2 > 0$. This is analogous to the case (3), section III of [26]. Under this, a stationary point is obtained by setting the update term in (4) to zero as $n \to \infty$, which leads to $E\big[\tau(\infty)\big(1-\tau(\infty)\big)^2\big]\triangle J_2 = E\big[\tau^2(\infty)\big(1-\tau(\infty)\big)\big]\triangle J_1$, where $\tau(\infty) = \lim_{n\to\infty} \tau(n)$. Assuming a negligibly small variance of $\tau(\infty)$, we have, $\tau(n) \to$ a constant (almost surely), as $n \to \infty$. One can then write from above, $\big(1 - E[\tau(\infty)]\big)\triangle J_2 = E[\tau(\infty)]\triangle J_1$, or equivalently,

$$E[\tau(\infty)] = \left(\frac{\triangle J_2}{\triangle J_1 + \triangle J_2}\right)_{1-\tau^+}^{\tau^+}. \tag{26}$$

It follows that

$$\begin{aligned} &\tau^+ \geq E[\tau(\infty)] > 0.5;\ \textit{if}\ J_{ex,1}(\infty) < J_{ex,2}(\infty) \\ &0.5 > E[\tau(\infty)] \geq 1 - \tau^+;\ \textit{if}\ J_{ex,1}(\infty) > J_{ex,2}(\infty). \end{aligned} \tag{27}$$

As proved in [26], this case is not sub-optimal. Rather it leads to

$$J_{ex}(\infty) \leq min\big(J_{ex,1}(\infty), J_{ex,2}(\infty)\big), \tag{28}$$

which means that in this case, the proposed convex combination works even better than each of its constituent filters.

3.4.3. Highly sparse systems

For highly sparse systems, i.e., for large values of M, two cases may arise as given below.

Case I : For $M_2 \leq M \leq L$, (i.e., assuming $M_2 < L$), for which, we have $J_{ex,2}(\infty) \leq J_{ex,12}(\infty) < J_{ex,1}(\infty)$, implying $\triangle J_1 > 0$ and $\triangle J_2 \leq 0$.

This is analogous to the case (2), section III of [26]. Equation (4) in this case leads to

$$\begin{aligned} E[a(n+1)] = E[a(n)] &+ \mu_a E\left[f_1\big(\tau(n)\big)\right]\triangle J_2 \\ &- \mu_a E\left[f_2\big(\tau(n)\big)\right]\triangle J_1, \end{aligned} \tag{29}$$

where $f_2\left(\tau(n)\right) = \tau^2(n)\big(1-\tau(n)\big) = f_1\big(1-\tau(n)\big)$. Like $f_1\big(\tau(n)\big)$, $f_2\big(\tau(n)\big) \geq f_2\big(1-\tau^+\big) \equiv f_1\big(\tau^+\big) = C, \forall \tau(n) \in [1-\tau^+, \tau^+]$. From the arguments used for the non-sparse case above, we have, $E[a(n+1)] \leq E[a(n)] - \mu_a C\big(\triangle J_1 - \triangle J_2\big)$, which is a decreasing sequence (since $\triangle J_1 > 0$ and $\triangle J_2 < 0$). This results in $\lim_{n\to\infty} E[a(n)]$ converging to $-a^+$, i.e., $\lim_{n\to\infty} a(n) = -a^+$ (almost surely), or, equivalently, $\lim_{n\to\infty} \tau(n) = 1 - \tau^+$ (almost surely) ≈ 0. The combination filter in this case will be converging to the Filter2, i.e., ZA-APA (in steady state) which is known to perform better for sparse systems.

Case II: For $M_2 > L$ and thus, $M_1 \leq M \leq L$, for which, we will have both $\triangle J_1 > 0$ and $\triangle J_2 > 0$.

This situation is same as the "less sparse" case considered above. The proposed combination in this case may perform better than each of the two constituent filters [26].

In Table 1, we summarize the above results. The table displays $J_{ex}(\infty)$ and $\tau(\infty)$ against different levels of sparsity of the system.

4. Numerical simulations

To validate the analytical results, a series of detailed simulations was conducted in the context of system identification. The proposed convex combination was simulated to identify a system with 64 coefficients that were chosen randomly from a zero-mean Gaussian distribution with variance 0.1 and 2.5×10^{-5} for active (i.e., taps taking high magnitude values) and inactive taps (i.e., taps taking very small values), respectively. Initially, the system was chosen to be highly non-sparse with the number of inactive taps M taken to be only 4 (equivalently, number of active taps taken to be 60). The simulation was conducted for a total duration of 20000 time steps. The sparsity level of the system was gradually increased after every 4000 time steps, leading to $M = 46, 52, 55$, and 63 after 4000, 8000, 12000, and 16000 time steps, respectively.

Simulations were conducted both for white and colored Gaussian input of unit variance, where a unity variance colored, Gaussian input $u(n)$ was generated by driving the following first order autoregressive (AR) model: $u(n) = \theta u(n-1) + \sqrt{1-\theta^2}z(n)$, $|\theta| < 1$, with a unit variance, white Gaussian input $z(n)$ (θ was taken to be 0.8 for the simulations). The observation noise $\vartheta(n)$ was taken to be i.i.d. Gaussian with mean zero and variance $\sigma_\vartheta^2 = 0.01$, implying a signal-to-noise ratio (SNR) of 20 dB. The constant ϵ in the APA update (used to avoid inversion of a rank deficient matrix) was taken to be 0.001, the projection order P was fixed at 2, the adaptation step size μ was set to 0.2 for both the algorithms and μ_a was kept at 50. The shrinkage parameter ρ was taken as 4×10^{-5} and 6.5×10^{-5} for white and colored input, respectively. The simulation results are displayed by plotting the EMSE $J_{ex}(n)$ (in dB) against the iteration index n, obtained by averaging $e_a^2(n)$ over 5000 experiments. The resulting plots, which are popularly called the learning curves, are shown in Figs. 2(a) and 3(a), respectively for white input and colored input cases, where we also plot the learning curves of the APA and the ZA-APA, i.e., $J_{ex,1}(n)$ and $J_{ex,2}(n)$ respectively, and the cross EMSE $J_{ex,12}(n)$ against the iteration index n. For comparative assessment, same identification exercise was also carried out by two QAPA-SSI filters [18], for which we chose $\alpha = 2 \times 10^{-5}$ and (same for both filters) and $\beta = 50$ and 0.05 with other parameters remaining same as used above. The EMSE of the two QAPA-SSI algorithms are also plotted in Figs. 2(a) and 3(a).

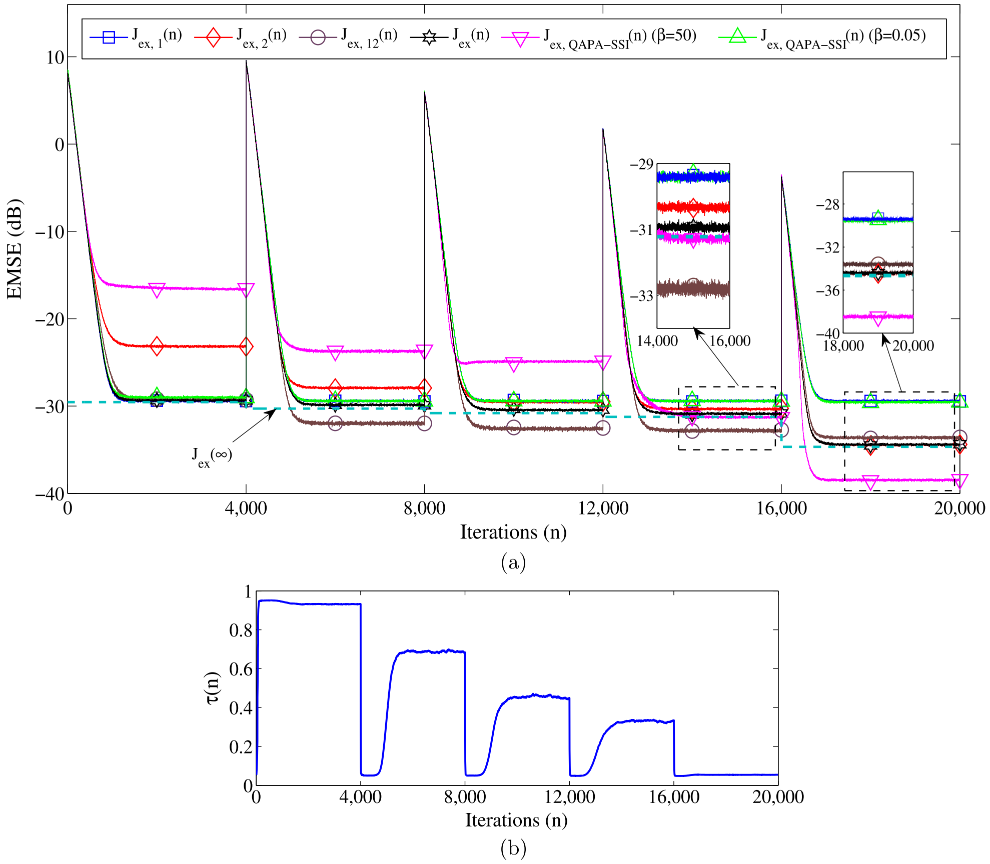


Fig. 2. (a) Learning curves of the APA, the ZA-APA and their convex combination for white input. Also shown are the learning curves of the QAPA-SSI algorithm for $\beta = 0.05$ and $\beta = 50$, and the cross EMSE $J_{ex,12}(n)$. (b) Evolution of the mixing parameter $\tau(n)$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

Several interesting observations can be made from Figs. 2(a) and 3(a) as described below.

1. For $0 \le n \le 4000$, the system is almost fully non-sparse and it is seen that in the steady state, $J_{ex,1}(n) \approx J_{ex,12}(n)$ with both $J_{ex,1}(n)$ and $J_{ex,12}(n)$ less than $J_{ex,2}(n)$. Also seen is that $J_{ex}(n) \approx J_{ex,1}(n)$, meaning, the convex combination switches to Filter1. This validates our conjecture above about fully non-sparse systems.
2. It is seen that for $4001 \le n \le 8000$, $J_{ex,1}(n) < J_{ex,2}(n)$, for $8001 \le n \le 12000$, $J_{ex,1}(n) \approx J_{ex,2}(n)$, while, for $12001 \le n \le 16000$, $J_{ex,1}(n) > J_{ex,2}(n)$ (all in the steady state). The first case corresponds to a moderately non-sparse system (Zone 1 of Fig. 1), the second case corresponds to the transition point M_1 of Fig. 1, while the third case corresponds to a moderately sparse system (Zone 2 of Fig. 1). In all the three cases, it is seen that $J_{ex,12}(n) < J_{ex,l}(n)$, $l = 1, 2$ (in the steady state). More interestingly, in all the three cases above, we have $J_{ex}(n) < J_{ex,l}(n)$, $l = 1, 2$. This validates our hypothesis that in case of both Zone 1 and Zone 2 of Fig. 1, as $J_{ex,12}(\infty) < J_{ex,l}(\infty)$, $l = 1, 2$, the proposed combination can converge to a filter that performs better than either of Filter1 and Filter2.
3. Lastly, for $16001 \le n \le 20000$, we have $J_{ex,1}(n) > J_{ex,12}(n) > J_{ex,2}(n)$ and $J_{ex}(n) = J_{ex,2}(n)$ (in the steady state). This corresponds to a highly sparse system (Zone 3 of Fig. 1). Again, this validates our claim that for a highly sparse system, the proposed combination will switch to Filter2.

Figs. 2(a) and 3(a) also present an interesting comparison of the proposed algorithm with the QAPA-SSI. It is seen that while the proposed method enjoys excellent performance at all levels of sparsity, the QAPA-SSI, depending on the value of β chosen, can perform at par or even better than the proposed method for certain sparsity level, while for other sparsity levels, its performance deteriorates vis-a-vis the proposed method. For example, as seen in Figs. 2(a) and 3(a), for $\beta = 50$, the QAPA-SSI achieves lesser EMSE than the proposed algorithm when the system is highly sparse (i.e., for $16001 \le n \le 20000$), while for all other sparsity levels, its EMSE is higher than that of the proposed method. Similarly, for $\beta = 0.05$, the QAPA-SSI performs at par with the proposed method when the system is highly non-sparse (i.e., for $0 \le n \le 4000$), while for all other levels of sparsity, the QAPA-SSI produces EMSE higher than that produced by the proposed algorithm.

We have also plotted the theoretical values of $J_{ex}(\infty)$ for white input as per (5) by horizontal dashed line in Fig. 2(a). It is easily seen that for each chosen value of M, the corresponding theoretical value matches the experimentally obtained value of $J_{ex}(\infty)$ almost exactly.

For this experiment, the time evolution of mixing coefficient $\tau(n)$ is presented in Figs. 2(b) and 3(b) for white and colored input, respectively. From Figs. 2(b) and 3(b), it can be observed that the mixing coefficient $\tau(n)$ converges toward τ^+ and $1 - \tau^+$ for non-sparse and highly sparse systems, respectively. Whereas, for the remaining three cases (i.e., moderately non-sparse system to moderately sparse system), it is observed that $\tau^+ \ge \tau(\infty) \ge 1 - \tau^+$.

Next, we evaluate the performance of the proposed convex combination for speech signal input. Here, the system is driven by a test speech signal obtained by repeating the speech waveform of Fig. 4 9 times. Using this speech input, the proposed adaptive convex combination was simulated for identifying a system whose impulse response undergoes some sudden changes and has

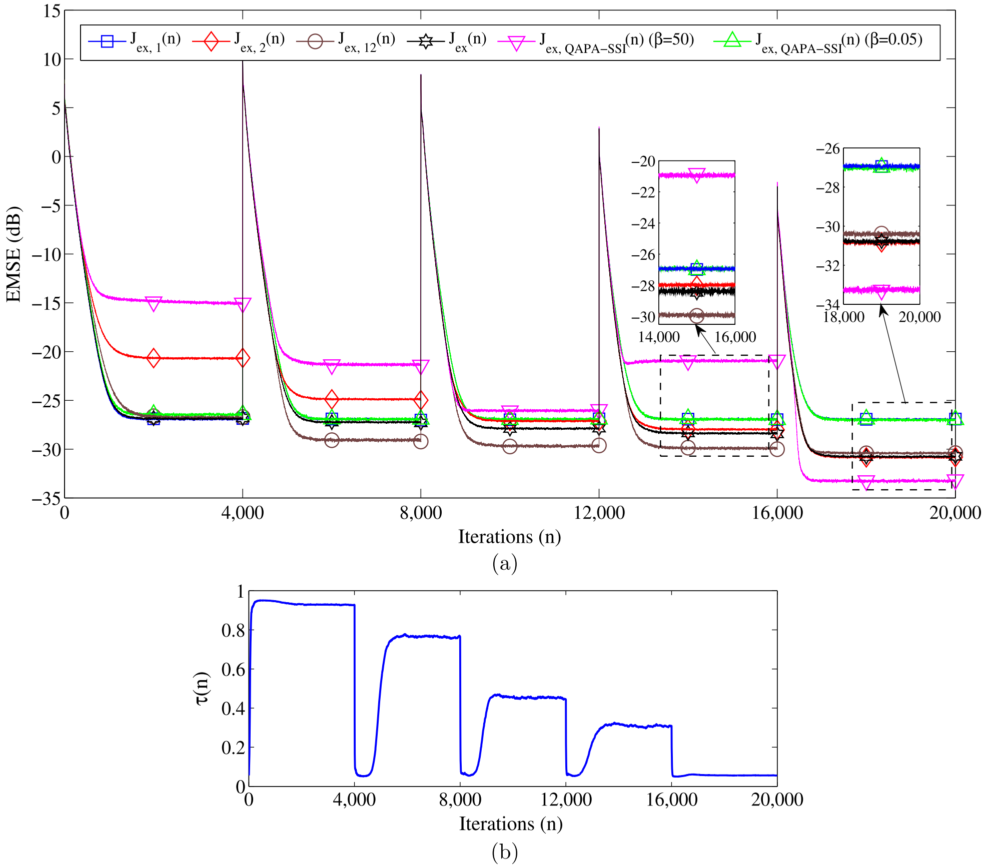


Fig. 3. (a) Learning curves of the APA, the ZA-APA and their convex combination for AR(1) input. Also shown are the learning curves of the QAPA-SSI algorithm for $\beta = 0.05$ and $\beta = 50$, and the cross EMSE $J_{ex,12}(n)$. (b) Evolution of the mixing parameter $\tau(n)$.

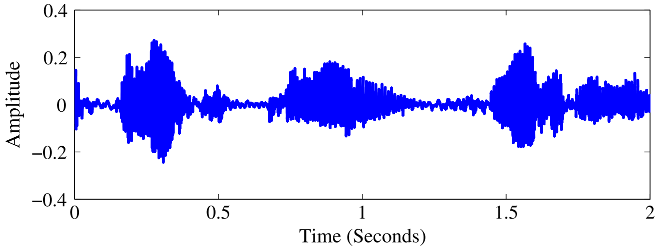


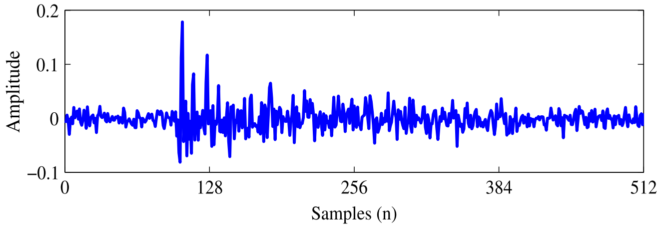
Fig. 4. Speech signal used to verify the performance (Identification and Tracking) of the proposed convex combination.

a duration of 0.064 sec (or equivalently, length $L = 512$ taps at 8 kHz sampling rate). Unlike the previous example, the channel coefficients in this case are not binary (i.e., having either large, or, zero magnitude), and instead, can assume any value with magnitude between zero and some upper limit. For such channels, we evaluate their sparsity by the following sparseness measure: $S_m = \frac{L}{L-\sqrt{L}}\left(1 - \frac{||\mathbf{w}_{opt}||_1}{\sqrt{L}||\mathbf{w}_{opt}||_2}\right)$ [28]. At first, the system was taken to be non sparse with the impulse response shown in Fig. 5(a), for which $S_m = 0.3460$. After 6 seconds, the system was changed to a measured acoustic echo path [29] shown in Fig. 5(b), which is less sparse or moderately sparse with the associated sparseness measure $S_m = 0.5560$. Finally, after 12 seconds, the system was taken to be the sparse network echo path as per ITU-T (International Telecommunication Union – Telecommunication standardization sector) G.168 recommendation [30], having a highly sparse impulse response as shown in Fig. 5(c), with $S_m = 0.8960$.

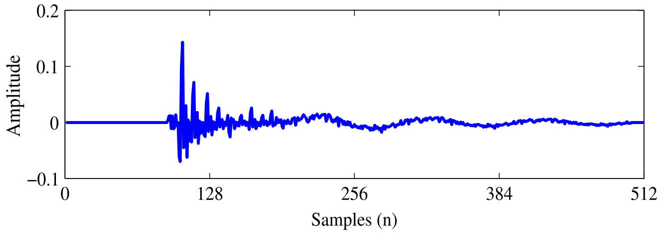
The observation noise $\vartheta(n)$ was taken to be i.i.d. Gaussian with mean zero and variance $\sigma_\vartheta^2 = 10^{-6}$ (i.e., echo signal-to-noise ratio (ENR) ≈ 20.5 dB [29]). Among other parameters, we chose $\epsilon = 0.01$, projection order $P = 4$, $\mu = 0.2$, $\mu_a = 5000$ and the shrinkage parameter $\rho = 1.2 \times 10^{-6}$. For QAPA-SSI, we chose $\alpha = 1 \times 10^{-7}$ and $\beta = \{1000, 10\}$ with other parameters remaining same as used for other schemes. The simulation results are displayed by plotting the MSD in dB vs time (in seconds), obtained by averaging over 50 experiments. The resulting plots are shown in Fig. 6(a).

From Fig. 6(a), it is observed that for speech input also, the simulation results validate the conjectures made by us regarding behavior of the proposed combination, as elaborated below.

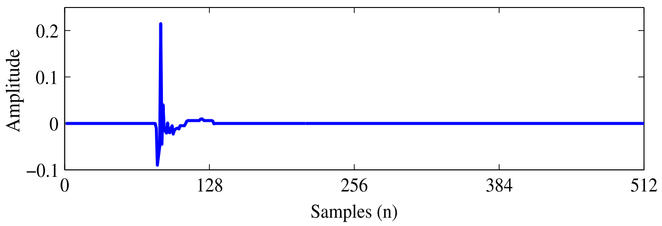
1. For the non-sparse system (i.e., for $0 \leq$ time $\leq 6s$), it is seen that in the steady state, MSD of the APA approximately equals the cross MSD, with both of them being less than the MSD of the ZA-APA. Also seen is that the MSD of the combination filter is virtually identical to the MSD of the APA, meaning, the convex combination in this case switches to Filter1. This validates our conjecture above about fully non-sparse systems.



(a) Non-sparse system, $S_m = 0.3486$.

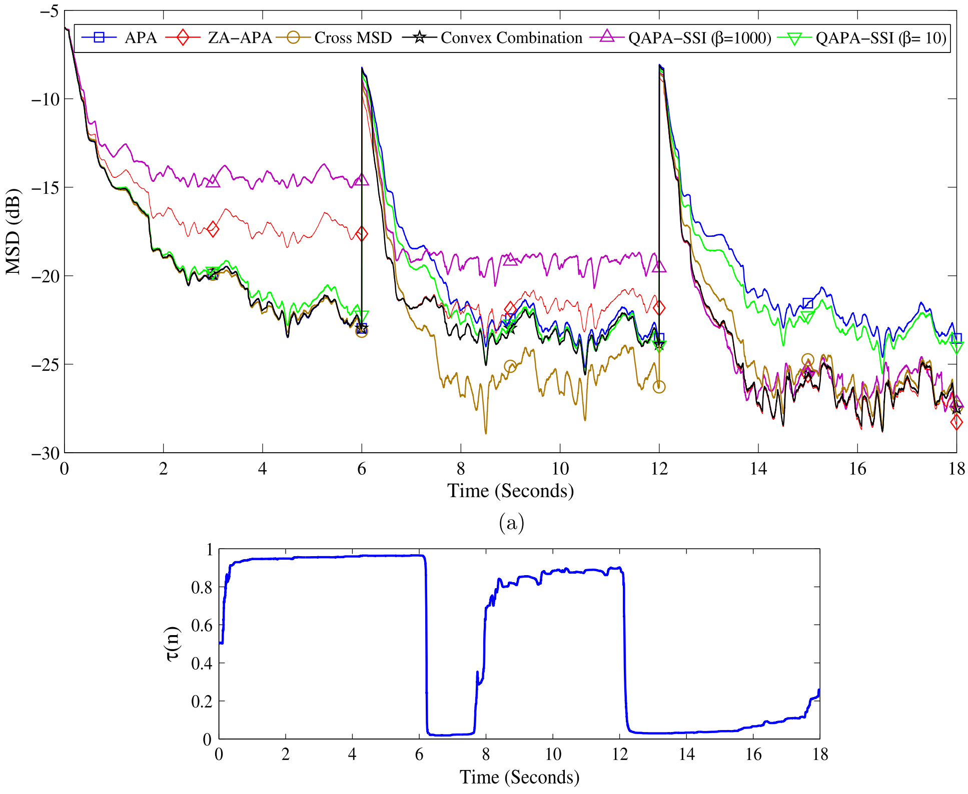


(b) Less/moderately sparse system, $S_m = 0.5560$.



(c) Highly sparse system, $S_m = 0.8960$.

Fig. 5. Impulse responses used in simulations.



(a)

(b)

Fig. 6. (a) Learning curves of the APA, the ZA-APA and their convex combination for speech signal input. Also shown are the learning curves of the QAPA-SSI algorithm for $\beta = 1000$ and $\beta = 10$, and the cross MSD. (b) Evolution of the mixing parameter $\tau(n)$.

2. Next, for the moderately sparse system (i.e., for $6s \le$ time $\le 12s$), the cross MSD falls below the MSD of both the individual filters (in the steady state). More interestingly, the MSD of the combination filter is seen to be less than the MSD of the individual filters. This validates our hypothesis that in case of Zone 2 of Fig. 1, the proposed combination performs better than either of the constituent filters.
3. Lastly, for highly sparse system (i.e., for $12s \le$ time $\le 18s$), both the MSD of the APA and the cross MSD are seen to be higher than the MSD of the ZA-APA, which also equals the MSD of the combination filter (in the steady state). Clearly, this validates our claim that for a highly sparse system, the proposed combination will favor Filter2.

Lastly, we have shown earlier (i.e., Figs. 2(a) and 3(a)) that both for white and correlated input, while the proposed method performs consistently well for all levels of sparsity, the QAPA-SSI, depending on the value of β chosen, can perform at par or even better than the proposed method for certain sparsity level, but its performance worsens vis-a-vis the proposed method as the sparsity level changes. From Fig. 6(a), it is seen that the same holds true even when the input is replaced by speech signal.

As we did in the earlier experiments, the time evolution of the mixing coefficient $\tau(n)$ for the simulation with speech input is presented in Fig. 6(b). From Fig. 6(b), it can be observed that the mixing coefficient $\tau(n)$ converges toward τ^+ for non-sparse system. In the case of measured acoustic echo path, which is moderately sparse, it is observed that $\tau^+ \ge \tau(\infty) \ge 1-\tau^+$. For network echo path, having highly sparse impulse response the mixing coefficient converges toward $1-\tau^+$.

5. Conclusions

A new APA based algorithm is proposed for identifying sparse systems with variable sparsity. The proposed method uses an adaptive convex combination of the standard APA and the sparsity aware ZA-APA. The algorithm adapts dynamically to the level of sparseness of the system. It also manifests superior convergence behavior with coloured input compared to its NLMS and LMS based counterparts because of its APA based constituents. A detailed performance analysis of the proposed combination is carried out, which reveals that while for highly sparse and highly non-sparse systems, the proposed combination converges respectively to the ZA-APA and APA (i.e., better of the two filters under the given levels of sparsity), for certain sparsity ranges, it leads to an overall filter that performs better than both the constituent filters. The claims made are validated by exhaustive simulation studies using white, correlated as well as speech inputs.

Appendix A. Proof of Theorem 1

Taking expectations on both sides of (14), assuming statistical independence between $\mathbf{w}_2(n)$ and $\mathbf{u}(n-j)$, $j = 0, 1, \cdots, P-1$ (i.e., generalizing the commonly used "independence assumption" [2]), and recalling that $\vartheta(n)$ is a zero-mean, i.i.d. random variable which is independent of $\mathbf{u}(m)$ for all n, m, one can obtain

$$E[\widetilde{\mathbf{w}}_2(n+1)] = \left(\mathbf{I}_L - \mu E\Big[\sum_{j\in J_n} \frac{\mathbf{u}(n-j)\mathbf{u}^T(n-j)}{\mathbf{u}^T(n-j)\mathbf{u}(n-j)}\Big]\right) E[\widetilde{\mathbf{w}}_2(n)] + \rho E\big[sgn\big(\mathbf{w}_2(n)\big)\big]. \tag{A.1}$$

Now, by invoking the angular discretization model of a random vector (15), the outer to inner product appearing in (A.1) can be simplified as:

$$\frac{\mathbf{u}(n-j)\mathbf{u}^T(n-j)}{\mathbf{u}^T(n-j)\mathbf{u}(n-j)} = \frac{s(n-j)r(n-j)\mathbf{v}(n-j)\mathbf{v}^T(n-j)s(n-j)r(n-j)}{s^2(n-j)r^2(n-j)\|\mathbf{v}(n-j)\|^2} = \mathbf{v}(n-j)\mathbf{v}^T(n-j), \tag{A.2}$$

where $\mathbf{v}(n-j) \in \{\mathbf{v}_0, \mathbf{v}_1, \cdots, \mathbf{v}_{L-1}\}$ (also note that the above result is independent of the norm of $\mathbf{u}(n-j)$).

Substituting (A.2), in (A.1), we have

$$E[\widetilde{\mathbf{w}}_2(n+1)] = \left(\mathbf{I}_L - \mu E\Big[\sum_{j\in J_n} \mathbf{v}(n-j)\mathbf{v}^T(n-j)\Big]\right) E[\widetilde{\mathbf{w}}_2(n)] + \rho E\big[sgn\big(\mathbf{w}_2(n)\big)\big]. \tag{A.3}$$

Note that while evaluating $E\big[\sum_{j\in J_n} \mathbf{v}(n-j)\mathbf{v}^T(n-j)\big]$, all possibilities of J_n are to be considered, e.g., $\{0\}, \{0,1\}, \{0,2\}, \{0,1,2\}, \cdots$.

Next, using the orthonormal basis $\{\mathbf{v}_0, \mathbf{v}_1, \cdots, \mathbf{v}_{L-1}\}$, we express $E[\widetilde{\mathbf{w}}_2(n)]$ as

$$E[\widetilde{\mathbf{w}}_2(n)] = \sum_{i=0}^{L-1} \alpha_{2,i}(n)\mathbf{v}_i, \quad \text{where, } \alpha_{2,i}(n) = \mathbf{v}_i^T E[\widetilde{\mathbf{w}}_2(n)].$$

Then, pre-multiplying both sides of (A.3) by $\mathbf{v}_i^T$, we obtain,

$$\alpha_{2,i}(n+1) = \left(\mathbf{v}_i^T - \mu E\Big[\mathbf{v}_i^T \sum_{j\in J_n} \mathbf{v}(n-j)\mathbf{v}^T(n-j)\Big]\right) E[\widetilde{\mathbf{w}}_2(n)] + \rho \mathbf{v}_i^T E\big[sgn\big(\mathbf{w}_2(n)\big)\big]. \tag{A.4}$$

From the orthonormality of $\mathbf{v}_k$'s, we have

$$\mathbf{v}_i^T \sum_{j\in J_n} \mathbf{v}(n-j)\mathbf{v}^T(n-j) = \begin{cases} \mathbf{v}_i^T, & \text{if } \exists j \in J_n \text{ so that } \mathbf{v}(n-j) \equiv \mathbf{v}_i \\ 0, & \text{otherwise.} \end{cases} \tag{A.5}$$

Substituting (A.5) in (A.4), we then have,

$$\alpha_{2,i}(n+1) = \big(1 - \mu\beta_i\big)\alpha_{2,i}(n) + \rho b_i(n), \tag{A.6}$$

where $b_i(n) = \mathbf{v}_i^T E\big[sgn\big(\mathbf{w}_2(n)\big)\big]$ and the term β_i in (A.6) is the probability of drawing an eigenvector $\mathbf{v}_i$ from the eigenvector set $\{\mathbf{v}_0, \mathbf{v}_1, \cdots, \mathbf{v}_{L-1}\}$ at least once in P trials after replacement.

To evaluate the non-linear term $b_i(n) = \mathbf{v}_i^T E\big[sgn\big(\mathbf{w}_2(n)\big)\big]$ in (A.6), we divide the coefficients of $\mathbf{w}_{opt}$ into two disjoint sets, namely, the set of non-zero (i.e., active) taps and the set of zero (i.e., inactive) taps, with NZ and Z denoting the sets of corresponding coefficient indices respectively, i.e., $w_{opt,i} = 0$ for $i \in Z$ and $w_{opt,i} \neq 0$ for $i \in NZ$. Also, we now invoke the assumption of white input, for which the eigenvectors $\{\mathbf{v}_0, \mathbf{v}_1, \cdots, \mathbf{v}_{L-1}\}$ become the trivial basis with $\mathbf{v}_i$ given by the i-th column of the $L \times L$ identity matrix $\mathbf{I}_L$, and thus, $\alpha_{2,i}(n) = E[\widetilde{w}_{2,i}(n)]$ and $b_i(n) = E\big[sgn\big(w_{2,i}(n)\big)\big]$. Also note that for white input, all the eigenvalues of $\mathbf{R}$ are same and given by σ_u^2. As a result, $p_i = \frac{\sigma_u^2}{Trace(\mathbf{R})} = \frac{1}{L}$: independent of i. As a result, β_i as used in (A.6) and given by $\beta_i = 1 - (1-p_i)^P$ is independent of the index i. In the following, we drop the subscript i from β_i, i.e., we replace β_i by $\beta = 1 - (1 - \frac{1}{L})^P$.

(a) Active taps ($i \in NZ$): We assume that each active tap has significant magnitude and also the variance $E[\widetilde{w}_{2,i}^2(n)]$ is small (especially near convergence). It is then reasonable to approximate $E\big[sgn\big(w_{2,i}(n)\big)\big]$ by $sgn\big(w_{opt,i}\big)$, i.e., in all trials, $w_{2,i}(n)$ assumes the same sign as that of $w_{opt,i}$. Substituting in (A.6), for $i \in NZ$, we then have,

$$E[\widetilde{w}_{2,i}(n+1)] = \big(1-\mu\beta\big)E[\widetilde{w}_{2,i}(n)] + \rho sgn\big(w_{opt,i}\big). \tag{A.7}$$

The above is a first order recursion of the type $\theta(n+1) = a\theta(n) + b$, which converges to $\frac{b}{1-a}$ for $|a| < 1$. This implies,

$$E\big[\widetilde{w}_{2,i}(n)\big]\Big|_{\infty} = \lim_{n\to\infty} E[\widetilde{w}_{2,i}(n)] = \frac{\rho}{\mu\beta} sgn(w_{opt,i}), \tag{A.8}$$

provided $-1 < 1-\mu\beta < 1$. As $0 < \beta < 1$, a sufficient condition for this is $0 < \mu < 2$. The result (16) follows trivially from (A.8) and the definition a).

(b) Inactive taps ($i \in Z$): On the other hand, for inactive taps, $w_{opt,i} = 0$, implying $\widetilde{w}_{2,i}(n) = -w_{2,i}(n)$. Substituting in (A.6), for $i \in Z$, we then have,

$$E[w_{2,i}(n+1)] = \big(1-\mu\beta\big)E[w_{2,i}(n)] - \rho E\big[sgn\big(w_{2,i}(n)\big)\big]. \tag{A.9}$$

The above is again a first order recursion of the form $\theta(n+1) = a\theta(n) + b(n)$, with $-\rho \le b(n) \le \rho$. For $|a| < 1$, as $n \to \infty$, $\theta(n)$ remains confined to the narrow strip $\Big[-\frac{\rho}{1-|a|}, \frac{\rho}{1-|a|}\Big]$. If $b(n)$ is a converging sequence, then we have, $\theta(\infty) = \frac{b(\infty)}{1-a}$. From this, we can write,

$$\overline{w}_{2,i}(\infty) = -\frac{\rho}{\mu\beta} E\big[sgn\big(w_{2,i}(n)\big)\big]\Big|_{\infty}. \tag{A.10}$$

Appendix B. Proof of Theorem 2

Post-multiplying the LHS and RHS of (14) by their respective transposes, taking expectation and using the above stated, generalized independence assumption, the recursion for $\mathbf{K}_2(n) \equiv E[\widetilde{\mathbf{w}}_2(n)\widetilde{\mathbf{w}}_2^T(n)]$ can be obtained as follows:

$$\begin{aligned}
\mathbf{K}_2(n+1) = \mathbf{K}_2(n) &- \mu E\Bigg[\sum_{j\in J_n} \frac{\mathbf{u}(n-j)\mathbf{u}^T(n-j)}{\mathbf{u}^T(n-j)\mathbf{u}(n-j)}\Bigg]\mathbf{K}_2(n) \\
&- \mu \mathbf{K}_2(n) E\Bigg[\sum_{m\in J_n} \frac{\mathbf{u}(n-m)\mathbf{u}^T(n-m)}{\mathbf{u}^T(n-m)\mathbf{u}(n-m)}\Bigg] \\
&+ \mu^2 E\Bigg[\Bigg(\sum_{j\in J_n} \frac{\mathbf{u}(n-j)\mathbf{u}^T(n-j)}{\mathbf{u}^T(n-j)\mathbf{u}(n-j)}\Bigg) \\
&\quad \times \mathbf{K}_2(n)\Bigg(\sum_{m\in J_n} \frac{\mathbf{u}(n-m)\mathbf{u}^T(n-m)}{\mathbf{u}^T(n-m)\mathbf{u}(n-m)}\Bigg)\Bigg] \\
&+ \mu^2 E\Bigg[\Bigg(\sum_{j\in J_n} \frac{\mathbf{u}(n-j)\vartheta(n-j)}{\mathbf{u}^T(n-j)\mathbf{u}(n-j)}\Bigg) \\
&\quad \times \Bigg(\sum_{m\in J_n} \frac{\mathbf{u}(n-m)\vartheta(n-m)}{\mathbf{u}^T(n-m)\mathbf{u}(n-m)}\Bigg)^T\Bigg] \\
&+ \rho E\Bigg[I_L - \mu\Bigg(\sum_{j\in J_n} \frac{\mathbf{u}(n-j)\mathbf{u}^T(n-j)}{\mathbf{u}^T(n-j)\mathbf{u}(n-j)}\Bigg)\Bigg] \\
&\quad \times E\big[\widetilde{\mathbf{w}}_2(n) sgn\big(\mathbf{w}_2^T(n)\big)\big] \\
&+ \rho E\big[sgn\big(\mathbf{w}_2(n)\big)\widetilde{\mathbf{w}}_2^T(n)\big] \\
&\quad \times E\Bigg[I_L - \mu\Bigg(\sum_{m\in J_n} \frac{\mathbf{u}(n-m)\mathbf{u}^T(n-m)}{\mathbf{u}^T(n-m)\mathbf{u}(n-m)}\Bigg)\Bigg] \\
&+ \rho^2 E\big[sgn\big(\mathbf{w}_2(n)\big) sgn\big(\mathbf{w}_2^T(n)\big)\big],
\end{aligned} \tag{B.1}$$

where the cross terms involving $\vartheta(n-j)$ are turn out to be zero as $\vartheta(m)$ is taken to be zero mean and statistically independent of $\mathbf{u}(n)$, for all n, m and, as $\mathbf{w}_2(n)$ is statistically independent of $\vartheta(n-j)$, $j = 0, 1, \cdots, P-1$. Using the whiteness of $\vartheta(n)$, (A.2) and the fact that $s(n)$, $r(n)$ and $\mathbf{v}(n)$ in (15) are statistically independent, we obtain,

$$\begin{aligned}
\mathbf{K}_2(n+1) = \mathbf{K}_2(n) &- \mu E\Big[\sum_{j\in J_n} \mathbf{v}(n-j)\mathbf{v}^T(n-j)\Big]\mathbf{K}_2(n) \\
&- \mu \mathbf{K}_2(n) E\Big[\sum_{m\in J_n} \mathbf{v}(n-m)\mathbf{v}^T(n-m)\Big] \\
&+ \mu^2 E\Big[\Big(\sum_{j\in J_n} \mathbf{v}(n-j)\mathbf{v}^T(n-j)\Big) \\
&\quad \times \mathbf{K}_2(n)\Big(\sum_{m\in J_n} \mathbf{v}(n-m)\mathbf{v}^T(n-m)\Big)\Big] \\
&+ \mu^2 \sigma_\vartheta^2 \sum_{j\in J_n} E\Big[\frac{1}{r^2(n-j)}\Big] \\
&\quad \times E\big[\mathbf{v}(n-j)\mathbf{v}^T(n-j)\big] \\
&+ \rho E\Big[I_L - \mu \sum_{j\in J_n} \mathbf{v}(n-j)\mathbf{v}^T(n-j)\Big] \\
&\quad \times E\big[\widetilde{\mathbf{w}}_2(n) sgn\big(\mathbf{w}_2^T(n)\big)\big] \\
&+ \rho E\big[sgn\big(\mathbf{w}_2(n)\big)\widetilde{\mathbf{w}}_2^T(n)\big] \\
&\quad \times E\Big[I_L - \mu \sum_{m\in J_n} \mathbf{v}(n-m)\mathbf{v}^T(n-m)\Big] \\
&+ \rho^2 E\big[sgn\big(\mathbf{w}_2(n)\big) sgn\big(\mathbf{w}_2^T(n)\big)\big].
\end{aligned} \tag{B.2}$$

Recalling that $\widetilde{\lambda}_{2,i}(n) = [\mathbf{\Lambda}_2(n)]_{i,i}$ where $\mathbf{\Lambda}_2(n) = \mathbf{V}^T\mathbf{K}_2(n)\mathbf{V}$, we pre-multiply and post-multiply both the LHS and the RHS of (B.2) by $\mathbf{v}_i^T$ and $\mathbf{v}_i$ (i.e., the i^{th} eigenvector of the $\mathbf{R}$) respectively, to obtain,

$$\begin{aligned}
\widetilde{\lambda}_{2,i}(n+1) = \widetilde{\lambda}_{2,i}(n) &- \mu E\Big[\mathbf{v}_i^T\Big(\sum_{j\in J_n} \mathbf{v}(n-j)\mathbf{v}^T(n-j)\Big)\Big]\mathbf{K}_2(n)\mathbf{v}_i \\
&- \mu \mathbf{v}_i^T \mathbf{K}_2(n) E\Big[\Big(\sum_{m\in J_n} \mathbf{v}(n-m)\mathbf{v}^T(n-m)\Big)\mathbf{v}_i\Big] \\
&+ \mu^2 E\Big[\mathbf{v}_i^T\Big(\sum_{j\in J_n} \mathbf{v}(n-j)\mathbf{v}^T(n-j)\Big) \\
&\quad \times \mathbf{K}_2(n)\Big(\sum_{m\in J_n} \mathbf{v}(n-m)\mathbf{v}^T(n-m)\Big)\mathbf{v}_i\Big] \\
&+ \mu^2 \sigma_\vartheta^2 \sum_{j\in J_n} E\Big[\frac{1}{r^2(n-j)}\Big] \\
&\quad \times E\big[\mathbf{v}_i^T\mathbf{v}(n-j)\mathbf{v}^T(n-j)\big]\mathbf{v}_i \\
&+ \rho E\Big[\mathbf{v}_i^T\Big(I_L - \mu \sum_{j\in J_n} \mathbf{v}(n-j)\mathbf{v}^T(n-j)\Big)\Big] \\
&\quad \times E\Big[\widetilde{\mathbf{w}}_2(n) sgn\big(\mathbf{w}_2^T(n)\big)\Big]\mathbf{v}_i \\
&+ \rho \mathbf{v}_i^T E\Big[sgn\big(\mathbf{w}_2(n)\big)\widetilde{\mathbf{w}}_2^T(n)\Big] \\
&\quad \times E\Big[\Big(I_L - \mu \sum_{m\in J_n} \mathbf{v}(n-m)\mathbf{v}^T(n-m)\Big)\mathbf{v}_i\Big] \\
&+ \rho^2 \mathbf{v}_i^T E\Big[sgn\big(\mathbf{w}_2(n)\big) sgn\big(\mathbf{w}_2^T(n)\big)\Big]\mathbf{v}_i.
\end{aligned} \tag{B.3}$$

To simplify the above, we recall from (A.5) that $E\big[\mathbf{v}_i^T \sum_{j\in J_n} \mathbf{v}(n-j)\mathbf{v}^T(n-j)\big] = \beta_i \mathbf{v}_i^T$, where $\beta_i =$ probability of drawing an eigenvector $\mathbf{v}_i$ from the eigenvector set $\{\mathbf{v}_0, \mathbf{v}_1, \cdots, \mathbf{v}_{L-1}\}$ at least once in P trials after replacement. Also, assuming that $\frac{1}{L}r^2(n)$ for any index n has a very small variance, which is potentially true for large filter order L, we can write, $E[\frac{1}{r^2(n-j)}] = \frac{1}{L}E[\frac{1}{\frac{1}{L}r^2(n-j)}] \approx \frac{1}{L}\frac{1}{E[\frac{1}{L}r^2(n-j)]} = \frac{1}{E[r^2(n-j)]} \equiv \frac{1}{Trace(\mathbf{R})}$. Making these substitutions above, we obtain,

$$\begin{aligned}\widetilde{\lambda}_{2,i}(n+1) &= \big(1-\mu(2-\mu)\beta_i\big)\widetilde{\lambda}_{2,i}(n) + \mu^2\sigma_\vartheta^2\frac{1}{\overline{r}^2}\beta_i \\ &\quad + \rho(1-\mu\beta_i)\Big(\mathbf{v}_i^T\big[\mathbf{\Phi}(n)+\mathbf{\Phi}^T(n)\big]\mathbf{v}_i\Big) \\ &\quad + \rho^2\Big[\mathbf{v}_i^T\mathbf{\Psi}(n)\mathbf{v}_i\Big],\end{aligned} \tag{B.4}$$

where we denote $Trace(\mathbf{R})$ by $\overline{r}^2$ for brevity, $\mathbf{\Phi}(n) = E\big[\widetilde{\mathbf{w}}_2(n)sgn\big(\mathbf{w}_2^T(n)\big)\big]$ and $\mathbf{\Psi}(n) = E\big[sgn\big(\mathbf{w}_2(n)sgn\big(\mathbf{w}_2^T(n)\big)\big]$.

At this stage, we invoke the given condition that the input is white. In such case, as seen earlier, the eigenvectors form the trivial basis of $\mathbb{R}^L$, with $\mathbf{v}_i$ given by the i-th column of the $L\times L$ identity matrix $\mathbf{I}_L$. Also, as seen before, p_i: probability of $\mathbf{v}(n) = \mathbf{v}_i$ is given by $\frac{1}{L}$ (i.e., same for all i) and thus $\beta_i = 1-(1-p_i)^P$ is independent of the index i (i.e., $\beta_i \equiv \beta$). Then, from (B.4), we have,

$$\begin{aligned}\widetilde{\lambda}_{2,i}(n+1) &= \big(1-\mu(2-\mu)\beta\big)\widetilde{\lambda}_{2,i}(n) + \mu^2\sigma_\vartheta^2\frac{1}{\overline{r}^2}\beta \\ &\quad + 2\rho(1-\mu\beta)E\big[\widetilde{w}_{2,i}(n)sgn\big(w_{2,i}(n)\big)\big] + \rho^2,\end{aligned} \tag{B.5}$$

where we have used the fact that $\big(sgn\big(w_{2,i}(n)\big)\big)^2 = 1$. Note that the recursion of $\widetilde{\lambda}_{2,i}(n)$ depends on $E\big[\widetilde{w}_{2,i}(n)sgn\big(w_{2,i}(n)\big)\big]$, which we evaluate separately for the active and the inactive taps as given below.

(a) Active taps ($i \in NZ$):

Since the active taps are having significantly large magnitudes and, near convergence, $|\widetilde{w}_{2,i}(n)| \ll |w_{opt,i}|$, as before, we may make the approximation $sgn(w_{2,i}(n)) = sgn(w_{opt,i})\ \forall i \in NZ$ (for large n). Therefore, $E\big[\widetilde{w}_{2,i}(n)sgn(w_{2,i}(n))\big] \approx sgn(w_{opt,i})E[\widetilde{w}_{2,i}(n)]$. From (A.8), for large n, $E[\widetilde{w}_{2,i}(n)] = \frac{\rho}{\mu\beta}sgn(w_{opt,i})$. Therefore, in the steady state, for $i \in NZ$,

$$E[\widetilde{w}_{2,i}(n)sgn(w_{2,i}(n))] \approx \frac{\rho}{\mu\beta}\big(sgn(w_{opt,i})\big)^2 = \frac{\rho}{\mu\beta}. \tag{B.6}$$

Using this result, from (B.5), for $i \in NZ$, we obtain,

$$\begin{aligned}\widetilde{\lambda}_{2,i}(n+1) &= \big(1-\mu(2-\mu)\beta\big)\widetilde{\lambda}_{2,i}(n) + \mu^2\sigma_\vartheta^2\frac{1}{\overline{r}^2}\beta \\ &\quad + \rho^2\frac{(2-\mu\beta)}{\mu\beta}.\end{aligned} \tag{B.7}$$

The above is a first order recursion of the type $\theta(n+1) = a\theta(n)+b$, which converges to $\frac{b}{1-a}$ for $|a|<1$. This implies,

$$\widetilde{\lambda}_{2,i}(\infty) = \frac{\mu}{2-\mu}\sigma_\vartheta^2\frac{1}{\overline{r}^2} + \rho^2\frac{(2-\mu\beta)}{\mu^2(2-\mu)\beta^2}, \tag{B.8}$$

provided $-1 < \big(1-\mu(2-\mu)\beta\big) < 1$. As $0<\beta<1$, a sufficient condition for this is $0<\mu<2$.

(b) Inactive taps ($i \in Z$):

On the other hand, for inactive taps, i.e., for $i \in Z$, $w_{opt,i} = 0$, and thus, we have $sgn(w_{2,i}(n)) = -sgn(\widetilde{w}_{2,i}(n))$, as $\widetilde{w}_{2,i}(n) = w_{opt,i} - w_{2,i}(n) = -w_{2,i}(n)$. We now invoke 'Price's theorem' [31], which states that if x, y are two zero mean, jointly Gaussian random variables, then,

$$E[xsgn(y)] = \frac{1}{\sigma_y}\sqrt{\frac{2}{\pi}}E[xy], \tag{B.9}$$

where $\sigma_y^2 = E[y^2]$ and σ_y is the positive square root of $E[y^2]$. In order to use this theorem, we first note that as ρ is very small, for large n (i.e., near convergence), (A.10) implies $E[w_{2,i}(n)] \approx 0$. It is also reasonable to assume that $w_{2,i}(n)$ is Gaussian distributed, as $w_{2,i}(n)$, through its weight update recursion, evolves from the data that is assumed to be Gaussian distributed. We can then write,

$$\begin{aligned}E\big[\widetilde{w}_{2,i}(n)sgn\big(w_{2,i}(n)\big)\big] &= -E\big[\widetilde{w}_{2,i}(n)sgn\big(\widetilde{w}_{2,i}(n)\big)\big] \\ &= -\sqrt{\frac{2}{\pi}}\sqrt{E[\widetilde{w}_{2,i}^2(n)]} \\ &= -\sqrt{\frac{2}{\pi}}\sqrt{\widetilde{\lambda}_{2,i}(n)}.\end{aligned} \tag{B.10}$$

Substituting the above in (B.5), for every $i \in Z$, we obtain

$$\begin{aligned}\widetilde{\lambda}_{2,i}(n+1) &= \big(1-\mu(2-\mu)\beta\big)\widetilde{\lambda}_{2,i}(n) + \mu^2\sigma_\vartheta^2\frac{1}{\overline{r}^2}\beta \\ &\quad - \sqrt{\frac{8}{\pi}}\rho(1-\mu\beta)\sqrt{\widetilde{\lambda}_{2,i}(n)} + \rho^2.\end{aligned} \tag{B.11}$$

Convergence of $\widetilde{\lambda}_{2,i}(n)$ then requires $-1 < \big(1-\mu(2-\mu)\beta\big) < 1$ and also $-1 < \sqrt{\frac{8}{\pi}}\rho(1-\mu\beta) < 1$. It is easy to see that both these conditions are satisfied simultaneously for $0<\mu<2$. Under this and letting $n\to\infty$ on both sides of (B.11), we then have

$$\begin{aligned}&\widetilde{\lambda}_{2,i}(\infty) + \sqrt{\frac{8}{\pi}}\rho\frac{(1-\mu\beta)}{\mu(2-\mu)\beta}\sqrt{\widetilde{\lambda}_{2,i}(\infty)} \\ &\quad - \left(\widetilde{\lambda}_{1,i}(\infty) + \frac{\rho^2}{\mu(2-\mu)\beta}\right) = 0,\end{aligned} \tag{B.12}$$

where $\widetilde{\lambda}_{1,i}(\infty) = \frac{\mu}{2-\mu}\sigma_\vartheta^2\frac{1}{\overline{r}^2}$, which follows by substituting $\rho = 0$ on the RHS of (B.5) (note that for $\rho = 0$, the ZA-APA turn out to be the simple APA). It then follows from (B.12) that $\sqrt{\widetilde{\lambda}_{2,i}(\infty)}$ is given by the real positive root of the following quadratic equation:

$$at^2 + bt + c = 0, \tag{B.13a}$$

where the coefficients are

$$\begin{aligned}a &= 1, \\ b &= \sqrt{\frac{8}{\pi}}\rho\frac{(1-\mu\beta)}{\mu(2-\mu)\beta}, \\ c &= -\left(\widetilde{\lambda}_{1,i}(\infty) + \frac{\rho^2}{\mu(2-\mu)\beta}\right).\end{aligned} \tag{B.13b}$$

It is easy to see that under $0<\mu<2$, c is always negative. Thus, the only positive root of (B.13a) is given by $\frac{-b+\sqrt{b^2-4ac}}{2}$. Substituting b and c from (B.13b), the positive root of (B.12) is given by,

$$\sqrt{\widetilde{\lambda}_{2,i}(\infty)} = \frac{\sqrt{\frac{2}{\pi}}\rho(1-\mu\beta)}{\mu(2-\mu)\beta}\Omega, \tag{B.14}$$

where $\Omega = \sqrt{1+\left(\left(\frac{\pi}{2}\frac{\mu(2-\mu)\beta}{(1-\mu\beta)^2}\right)\left(1+\frac{\mu^2\beta\sigma_\vartheta^2}{\overline{r}^2\rho^2}\right)\right)} - 1$.

Squaring both sides of (B.14), the steady state mean square deviation of a single inactive tap is obtained as follows:

$$\begin{aligned}\widetilde{\lambda}_{2,i}(\infty)=\,&\widetilde{\lambda}_{1,i}(\infty)\\&+\frac{\rho^2}{\big(\mu(2-\mu)\beta\big)^2}\left(-\frac{4}{\pi}(1-\mu\beta)^2\Omega+\mu(2-\mu)\beta\right),\end{aligned}\tag{B.15}$$

In steady state, the EMSE of the ZA-APA is then given as follows:

$$\begin{aligned}J_{ex,2}(\infty)&=\sum_{i\in NZ}\lambda_i\widetilde{\lambda}_{2,i}(\infty)+\sum_{i\in Z}\lambda_i\widetilde{\lambda}_{2,i}(\infty)\\&=\sigma_u^2\left((L-M)\widetilde{\lambda}_{2,i\in NZ}(\infty)+M\widetilde{\lambda}_{2,i\in Z}(\infty)\right),\end{aligned}\tag{B.16}$$

where we have used M to denote the number of inactive taps.

Substituting (B.8) and (B.14) in (B.16), we obtain (19).

Appendix C. Proof of Theorem 3

Post-multiplying the LHS and RHS of $\widetilde{\mathbf{w}}_1(n+1)$ (obtained by setting $\rho=0$ in (14)) by the transposes of the LHS and RHS of (14) respectively, taking expectation, as earlier, using the aforesaid generalized independence assumption for both $\mathbf{w}_1(n)$ and $\mathbf{w}_2(n)$, and whiteness of $\vartheta(n)$, and also using the discretization model (15), the recursion for $\mathbf{K}_{12}(n)\equiv E[\widetilde{\mathbf{w}}_1(n)\widetilde{\mathbf{w}}_2^T(n)]$ can be obtained as follows:

$$\begin{aligned}\mathbf{K}_{12}(n+1)=\mathbf{K}_{12}(n)&-\mu E\Big[\sum_{j\in J_n}\mathbf{v}(n-j)\mathbf{v}^T(n-j)\Big]\mathbf{K}_{12}(n)\\&-\mu\mathbf{K}_{12}(n)E\Big[\sum_{m\in J_n}\mathbf{v}(n-m)\mathbf{v}^T(n-m)\Big]\\&+\mu^2E\Big[\Big(\sum_{j\in J_n}\mathbf{v}(n-j)\mathbf{v}^T(n-j)\Big)\\&\quad\times\mathbf{K}_{12}(n)\Big(\sum_{m\in J_n}\mathbf{v}(n-m)\mathbf{v}^T(n-m)\Big)\Big]\\&+\mu^2\sigma_\vartheta^2\sum_{j\in J_n}E\Big[\frac{1}{r^2(n-j)}\Big]\\&\quad\times E\big[\mathbf{v}(n-j)\mathbf{v}^T(n-j)\big]\\&+\rho E\Big[I_L-\mu\sum_{j\in J_n}\mathbf{v}(n-j)\mathbf{v}^T(n-j)\Big]\\&\quad\times E\big[\widetilde{\mathbf{w}}_1(n)sgn\big(\mathbf{w}_2^T(n)\big)\big],\end{aligned}\tag{C.1}$$

where the out-to-inner product ratios are replaced using (A.2) and the fact that $s(n)$, $r(n)$ and $\mathbf{v}(n)$ are statistically independent. Also, the cross terms involving $\vartheta(n-j)$ turn out to be zero as $\vartheta(m)$ is taken to be zero mean and independent of $\mathbf{u}(n)$, for all n, m and, as both $\mathbf{w}_1(n)$ and $\mathbf{w}_2(n)$ are statistically independent of $\vartheta(n-j)$, $j=0,1,\cdots,P-1$. Recalling that $\widetilde{\lambda}_{12,i}(n)=[\mathbf{\Lambda}_{12}(n)]_{i,i}$ where $\mathbf{\Lambda}_{12}(n)=\mathbf{V}^T\mathbf{K}_{12}(n)\mathbf{V}$, we pre-multiply and post-multiply both the LHS and the RHS of (C.1) by $\mathbf{v}_i^T$ and $\mathbf{v}_i$ (i.e., the i^{th} eigenvector of the $\mathbf{R}$) respectively, to obtain:

$$\begin{aligned}\widetilde{\lambda}_{12,i}(n+1)=\widetilde{\lambda}_{12,i}(n)&\\-\mu E\Big[\mathbf{v}_i^T\Big(\sum_{j\in J_n}\mathbf{v}(n-j)\mathbf{v}^T(n-j)\Big)\Big]\mathbf{K}_{12}(n)\mathbf{v}_i&\\-\mu\mathbf{v}_i^T\mathbf{K}_{12}(n)E\Big[\Big(\sum_{m\in J_n}\mathbf{v}(n-m)\mathbf{v}^T(n-m)\Big)\mathbf{v}_i\Big]&\\+\mu^2E\Big[\mathbf{v}_i^T\Big(\sum_{j\in J_n}\mathbf{v}(n-j)\mathbf{v}^T(n-j)\Big)&\\\times\mathbf{K}_{12}(n)\Big(\sum_{m\in J_n}\mathbf{v}(n-m)\mathbf{v}^T(n-m)\Big)\mathbf{v}_i\Big]&\\+\mu^2\sigma_\vartheta^2\sum_{j\in J_n}E\Big[\frac{1}{r^2(n-j)}\Big]&\\\times E\Big[\mathbf{v}_i^T\mathbf{v}(n-j)\mathbf{v}^T(n-j)\Big]\mathbf{v}_i&\\+\rho E\Big[\mathbf{v}_i^T\Big(I_L-\mu\sum_{j\in J_n}\mathbf{v}(n-j)\mathbf{v}^T(n-j)\Big)\Big]&\\\times E\big[\widetilde{\mathbf{w}}_1(n)sgn\big(\mathbf{w}_2^T(n)\big)\big]\mathbf{v}_i.&\end{aligned}\tag{C.2}$$

To simplify the above, we recall the fact that $E\big[\mathbf{v}_i^T\sum_{j\in J_n}\mathbf{v}(n-j)\mathbf{v}^T(n-j)\big]=\beta_i\mathbf{v}_i^T$ and the assumption $E[\frac{1}{r^2(n-j)}]\approx\frac{1}{E[r^2(n-j)]}\equiv\frac{1}{Trace(\mathbf{R})}$. Making these substitutions above and denoting $Trace(\mathbf{R})$ by $\bar{r}^2$, we obtain,

$$\begin{aligned}\widetilde{\lambda}_{12,i}(n+1)=\,&\big(1-\mu(2-\mu)\beta_i\big)\widetilde{\lambda}_{12,i}(n)+\mu^2\sigma_\vartheta^2\frac{1}{\bar{r}^2}\beta_i\\&+\rho(1-\mu\beta_i)\mathbf{v}_i^TE\Big[\widetilde{\mathbf{w}}_1(n)sgn\big(\mathbf{w}_2^T(n)\big)\Big]\mathbf{v}_i.\end{aligned}\tag{C.3}$$

At this stage, we invoke the white input condition, for which, as seen earlier, the eigenvector $\mathbf{v}_i$ is given by the i-th column of the $L\times L$ identity matrix $\mathbf{I}_L$. Also, as seen before, p_i: probability of $\mathbf{v}(n)=\mathbf{v}_i$ is given by $\frac{1}{L}$ (i.e., same for all i) for white input and thus β_i is independent of the index i (i.e., $\beta_i=\beta$). Then, from (C.3), we have,

$$\begin{aligned}\widetilde{\lambda}_{12,i}(n+1)=\,&\big(1-\mu(2-\mu)\beta\big)\widetilde{\lambda}_{12,i}(n)+\mu^2\sigma_\vartheta^2\frac{1}{\bar{r}^2}\beta\\&+\rho(1-\mu\beta)E\big[\widetilde{w}_{1,i}(n)sgn\big(w_{2,i}(n)\big)\big].\end{aligned}\tag{C.4}$$

Note that the recursion of $\widetilde{\lambda}_{12,i}(n)$ depends on $\mathbf{v}_i^TE\big[\widetilde{\mathbf{w}}_1(n)sgn\big(\mathbf{w}_2^T(n)\big)\big]\mathbf{v}_i$, which we evaluate separately for the active and the inactive taps as given below.

(a) Active taps ($i\in NZ$):

Since the active taps are having significantly large magnitudes and, near convergence, $|\widetilde{w}_{2,i}(n)|\ll|w_{opt,i}|$, as before, we may make the approximation $sgn(w_{2,i}(n))=sgn(w_{opt,i})$ $\forall i\in NZ$ (for large n). Therefore, $E\big[\widetilde{w}_{1,i}(n)sgn\big(w_{2,i}(n)\big)\big]\approx sgn(w_{opt,i})E[\widetilde{w}_{1,i}(n)]$. Now, using the fact that for large n, $E\big[\widetilde{w}_{1,i}(n)\big]=0$ [8] (also can be obtained by setting $\rho=0$ in (A.8)), in steady state, for $i\in NZ$,

$$E[\widetilde{w}_{1,i}(n)sgn(w_{2,i}(n))]\approx sgn(w_{opt,i})E[\widetilde{w}_{1,i}(n)]=0.\tag{C.5}$$

Substituting this in (C.4), for $i\in NZ$, we obtain,

$$\widetilde{\lambda}_{12,i}(n+1)=\big(1-\mu(2-\mu)\beta\big)\widetilde{\lambda}_{12,i}(n)+\mu^2\sigma_\vartheta^2\frac{1}{\bar{r}^2}\beta.\tag{C.6}$$

The above recursion is same as that of $\lambda_{1,i}(n)$, i.e., MSD of i-th coefficient of Filter1 [8] (which can be verified by setting $\rho=0$ in (B.7)), and a sufficient condition for its convergence is $0<\mu<2$. Under this, in steady state, for $i\in NZ$,

$$\widetilde{\lambda}_{12,i}(\infty)=\frac{\mu}{2-\mu}\sigma_\vartheta^2\frac{1}{\bar{r}^2}=\widetilde{\lambda}_{1,i}(\infty),\tag{C.7}$$

where $\widetilde{\lambda}_{1,i}(\infty)=\lim_{n\to\infty}\widetilde{\lambda}_{1,i}(n)$ is the steady state MSD of i-th coefficient of Filter1 [8].

(b) Inactive taps ($i\in Z$):

On the other hand, for inactive taps, i.e., for $i\in Z$, $w_{opt,i}=0$, implying $sgn\big(w_{2,i}(n)\big)=-sgn\big(\widetilde{w}_{2,i}(n)\big)$. Then, as ρ is very small, from (A.10), in the steady state, $E\big[w_{2,i}(n)\big]=-E\big[\widetilde{w}_{2,i}(n)\big]\approx 0$.

Also, in the steady state, we have $E\left[\widetilde{w}_{1,i}(n)\right] = 0$. Assuming $w_{1,i}(n)$ and $w_{2,i}(n)$ to be jointly Gaussian (which is reasonable as both are generated from the Gaussian distributed $u(n)$ and $\vartheta(n)$), we can write using the aforesaid Price's theorem [31], for $i \in Z$,

$$\begin{aligned} E\left[\widetilde{w}_{1,i}(n)sgn\left(w_{2,i}(n)\right)\right] &= -E\left[\widetilde{w}_{1,i}(n)sgn\left(\widetilde{w}_{2,i}(n)\right)\right] \\ &= -\sqrt{\frac{2}{\pi\sigma^2_{w_{2,i}}}}\widetilde{\lambda}_{12,i}(n) \\ &= -\sqrt{\frac{2}{\pi}}\frac{\widetilde{\lambda}_{12,i}(n)}{\sqrt{\widetilde{\lambda}_{2,i}(n)}}. \end{aligned} \tag{C.8}$$

With these results for $i \in Z$, from (C.4) we obtain

$$\begin{aligned} \widetilde{\lambda}_{12,i}(n+1) &= \left(1-\mu(2-\mu)\beta\right)\widetilde{\lambda}_{12,i}(n) + \mu^2\sigma^2_{\vartheta}\frac{1}{\bar{r}^2}\beta \\ &\quad - \sqrt{\frac{2}{\pi}}\rho(1-\mu\beta)\frac{\widetilde{\lambda}_{12,i}(n)}{\sqrt{\widetilde{\lambda}_{2,i}(n)}}. \end{aligned} \tag{C.9}$$

Convergence of $\widetilde{\lambda}_{12,i}(n)$ then requires $-1 < \left(1-\mu(2-\mu)\beta\right) < 1$ and also $-1 < \sqrt{\frac{2}{\pi\widetilde{\lambda}_{2,i}(n)}}\rho(1-\mu\beta) < 1$ simultaneously. It is easy to see that these are satisfied simultaneously for $0 < \mu < 2$, under sufficiently small value of ρ. Under this and letting $n \to \infty$ on both sides of (C.9), we finally have,

$$\begin{aligned} \widetilde{\lambda}_{12,i}(\infty) &= \frac{\mu^2\sigma^2_{\vartheta}\beta\frac{1}{\bar{r}^2}}{\mu(2-\mu)\beta + \rho(1-\mu\beta)\sqrt{\frac{2}{\pi\widetilde{\lambda}_{2,i}(\infty)}}} \\ &= \widetilde{\lambda}_{1,i}(\infty)\left(\frac{\sqrt{\widetilde{\lambda}_{2,i}(\infty)}}{\sqrt{\widetilde{\lambda}_{2,i}(\infty)} + \sqrt{\frac{2}{\pi}}\frac{\rho(1-\mu\beta)}{\mu(2-\mu)\beta}}\right). \end{aligned} \tag{C.10}$$

Substituting $\sqrt{\widetilde{\lambda}_{2,i}(\infty)}$ from (B.14) and after some simplifications, we obtain,

$$\widetilde{\lambda}_{12,i}(\infty) = \widetilde{\lambda}_{1,i}(\infty)\left(1 - \frac{1}{\Omega+1}\right). \tag{C.11}$$

In steady state, the cross EMSE of the APA and the ZA-APA is then given as follows:

$$\begin{aligned} J_{ex,12}(\infty) &= \sum_{i\in NZ}\lambda_i\widetilde{\lambda}_{12,i}(\infty) + \sum_{i\in Z}\lambda_i\widetilde{\lambda}_{12,i}(\infty) \\ &= \sigma^2_u\left((L-M)\widetilde{\lambda}_{12,i\in NZ}(\infty) + M\widetilde{\lambda}_{12,i\in Z}(\infty)\right). \end{aligned} \tag{C.12}$$

Substituting (C.7) and (C.11) in (C.12), we get (21).

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