Improving the Bound on the RIP Constant in Generalized Orthogonal Matching Pursuit

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***Abstract*—The generalized Orthogonal Matching Pursuit (gOMP) is a recently proposed compressive sensing greedy recovery algorithm which generalizes the OMP algorithm by selecting $N(\geq 1)$ atoms in each iteration. In this letter, we demonstrate that the gOMP can successfully reconstruct a K-sparse signal from a compressed measurement $\mathbf{y} = \mathbf{\Phi x}$ by a maximum of K iterations if the sensing matrix $\mathbf{\Phi}$ satisfies the Restricted Isometry Property (RIP) of order NK, with the RIP constant δ_{NK} satisfying $\delta_{NK} < \frac{\sqrt{N}}{\sqrt{K}+2\sqrt{N}}$. The proposed bound is an improvement over the existing bound on δ_{NK}. We also show that by increasing the RIP order just by one (i.e., $NK+1$ from NK), it is possible to refine the bound further to $\delta_{NK+1} < \frac{\sqrt{N}}{\sqrt{K}+\sqrt{N}}$, which is consistent (for $N = 1$) with the near optimal bound on δ_{K+1} in OMP.**

***Index Terms*—Compressive sensing, orthogonal matching pursuit, restricted isometry property, sensing matrix.**

I. Introduction

THE generalized orthogonal matching pursuit (gOMP) [1] is a generalization of the well known compressed sensing [2], [3] greedy recovery algorithm called orthogonal matching pursuit (OMP) [4]. Both the gOMP and the OMP algorithms try to obtain the sparsest solution to an underdetermined set of equations given as

$$\mathbf{y} = \mathbf{\Phi x},$$

where $\mathbf{\Phi}$ is a $m \times n$ $(m \ll n)$ real valued, sensing matrix and y is a $m \times 1$ real valued observation vector. It is assumed that the sparsest solution to the above system is K-sparse, i.e., not more than K (for some minimum K, $K > 0$) elements of $\mathbf{x}$ are non-zero and also that the sparsest solution is unique, which means that every $2K$ columns of $\mathbf{\Phi}$ are linearly independent [5]. Greedy approaches like the OMP, gOMP, and also the orthogonal least square (OLS) [6], the compressive sampling matching pursuit (CoSaMP) [7], the subspace pursuit (SP) [8] etc recover the K-sparse signal by iteratively constructing the support set of the sparse signal (i.e., index of non-zero elements in the sparse vector) by some greedy principles. Convergence of these iterative procedures in finite number of steps requires the matrix $\mathbf{\Phi}$ to satisfy the so-called "Restricted Isometry Property (RIP)" [9] of appropriate order as given below.

Definition 1: A matrix $\mathbf{\Phi}^{m \times n} (m < n)$ is said to satisfy the RIP of order K if there exists a "Restricted Isometry Constant" $\delta_K \in (0,1)$ so that

$$(1-\delta_K)\|\mathbf{x}_2^2 \leqslant \|\mathbf{\Phi x}\|_2^2 \leqslant (1+\delta_K)\|\mathbf{x}\|_2^2 \tag{1}$$

for all K-sparse $\mathbf{x}$. The constant δ_K is taken as the smallest number from (0, 1) for which the RIP is satisfied.

It is easy to see that if $\mathbf{\Phi}$ satisfies RIP of order K, then it also satisfies RIP for any order L where $L < K$ and that $\delta_K \geq \delta_L$. Simple choice of a random matrix for $\mathbf{\Phi}$ can make it satisfy the RIP condition with high probability [9].

Convergence of the above stated greedy algorithms is usually established by imposing certain upper bounds on the RIP constant δ_K as a sufficient condition. In the case of gOMP, such bound is given by $\delta_{NK} < \frac{\sqrt{N}}{\sqrt{K}+3\sqrt{N}}$ [1], where N is the number of columns of $\mathbf{\Phi}$ (also called atoms) the algorithm selects in each iteration. It is, however, observed that this bound is overly restrictive as it is obtained by applying a series of inequalities that makes it progressively tighter. In this letter, by following alternate algebraic manipulations, we improve this bound to $\delta_{NK} < \frac{\sqrt{N}}{\sqrt{K}+2\sqrt{N}}$. Further, we show that by increasing the RIP order just by one (i.e., $NK+1$ from NK), it is possible to refine[1] the bound further to $\delta_{NK+1} < \frac{\sqrt{N}}{\sqrt{K}+\sqrt{N}}$. The latter is interestingly seen to be consistent (for $N = 1$) with the near optimal upper bound $\frac{1}{\sqrt{K+1}}$ of δ_{K+1} in the OMP algorithm [10].

II. Notations and a Brief Review of the gOMP Algorithm

A. Notations

For the sake of uniformity, we follow the same notations as used in [1]. Let Z denote the index set $\{1, 2, \cdots, n\}$ and $\boldsymbol{\phi}_i, i \in Z$ denote the i-th column of $\mathbf{\Phi}$. The matrix $\mathbf{\Phi}_A$ represents the sub-matrix of $\mathbf{\Phi}$ with columns indexed by the elements present in set $A \subset Z$. Similarly $\mathbf{x}_A$ represents the sub-vector of $\mathbf{x}$ with elements chosen as per the indices given in A. By T, we denote the true support set of $\mathbf{x}$, meaning $|T| \leq K$ where $|.|$ denotes the cardinality of the set ".", and by Λ^k, we denote the estimated support set after k iterations of the algorithm. The notation $T - \Lambda^k$ indicates a set with elements belonging to T but not contained in Λ^k, i.e., $T - \Lambda^k = T \cap \overline{\Lambda^k}$. The pseudo-inverse of $\mathbf{\Phi}_A$ is denoted by $\mathbf{\Phi}_A^\dagger$, i.e., $\mathbf{\Phi}_A^\dagger = \left(\mathbf{\Phi}_A^t \mathbf{\Phi}_A\right)^{-1} \mathbf{\Phi}_A^t$, where it

[1]Note that as the upper bound of the RIP constant increases, one can choose larger values for the RIP constant than possible earlier, which in turn permits further compression of the number of measurements (i.e., m) without disturbing the high probability of satisfaction of the RIP by the sensing matrix $\boldsymbol{\phi}$ [11].

Manuscript received June 11, 2013; revised August 06, 2013; accepted August 21, 2013. Date of publication August 29, 2013; date of current version September 16, 2013. The associate editor coordinating the review of this manuscript and approving it for publication was Prof. Alexandros G. Dimakis.

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Digital Object Identifier 10.1109/LSP.2013.2279977

TABLE I
gOMP ALGORITHM

Input: measurement $\mathbf{y} \in \mathbb{R}^m$, sensing matrix $\mathbf{\Phi}^{m\times n}$, sparsity K, number of indices to be chosen per iteration N
Initialization: counter k=0, residue $\mathbf{r}^0$=$\mathbf{y}$, estimated support set $\Lambda^k = \emptyset$
While k<K and $\|\|\mathbf{r}^k\|\|_2 > 0$
Identification: h^{k+1}=Set of indices corresponding to the N largest entries in $\|\mathbf{\Phi}^t\mathbf{r}^k\|$. $(NK \leq m)$
Augment: $\Lambda^{k+1} = \Lambda^k \cup \{h^{k+1}\}$
Estimate: $\mathbf{b}_{\Lambda^{k+1}} = \arg\min_{\mathbf{z}} \|\|y - \mathbf{\Phi}_{\Lambda^{k+1}}\mathbf{z}\|\|_2$
Update: $\mathbf{r}^{k+1} = y - \mathbf{\Phi}_{\Lambda^{k+1}}\mathbf{b}_{\Lambda^{k+1}}$
k=k+1
End While
Output: $\hat{\mathbf{x}} = \arg\min_{\mathbf{u}:supp(\mathbf{u})=\Lambda^k} \|\|y - \mathbf{\Phi u}\|\|_2$
Note : For the noise free case, if $\mathbf{\Phi}$ satisfies RIP as given either in this paper or in [1], we will have $\hat{\mathbf{x}} = \mathbf{x}$ (i.e., the true K-sparse solution).

is assumed that $\mathbf{\Phi}_A$ has full column rank $(|A| < m)$. The two matrices, $\mathbf{P}_A = \mathbf{\Phi}_A\mathbf{\Phi}_A^\dagger$ and $\mathbf{P}_A^\perp = \mathbf{I} - \mathbf{P}_A$ denote two orthogonal projection operators which project a given vector orthogonally on the column space of $\mathbf{\Phi}_A$ and on its orthogonal complement respectively. Similarly, $\mathbf{A}_I = \mathbf{P}_I^\perp\mathbf{\Phi}$ is a matrix with each column denoting the projection error vector related to the orthogonal projection of the corresponding column of $\mathbf{\Phi}$ on the column space of $\mathbf{\Phi}_I$ for some $I \subset Z$. Lastly, we use "t" in the superscript to denote matrix/vector transposition and by "$span(.)$", we denote the subspace spanned by the columns of the matrix ".".

For convenience of presentation, we also follow the following convention: we use the notation $\overset{L1}{=}$ or $\overset{(1)}{=}$ or $\overset{D1}{=}$ or $\overset{T1}{=}$ to indicate that the equality "=" follows from Lemma 1/Equation (1)/Definition 1/Theorem 1 respectively (same for inequalities).

B. *A Brief Review of the* gOMP *Algorithm*

The gOMP algorithm is listed in Table I. At the k-th iteration $(k = 0, 1, \cdots, K-1)$, it evaluates the correlations between an available residual $\mathbf{r}^k$ and the atoms of $\mathbf{\Phi}$, and picks up the N largest correlated atoms. The corresponding indices are appended to an estimated support set Λ^k to yield Λ^{k+1} $(\Lambda^0 = \emptyset)$ and the residual is updated as the orthogonal projection error $\mathbf{r}^{k+1} = \mathbf{P}_{\Lambda^{k+1}}^\perp\mathbf{y}$ (which implies that at any k-th step of iteration, $\mathbf{r}^{k+1}$ is orthogonal to each column of $\mathbf{\Phi}_{\Lambda^{k+1}}$). It is shown in [1] that under the sufficient condition $\delta_{NK} < \frac{\sqrt{N}}{\sqrt{K}+3\sqrt{N}}$, the algorithm converges in a maximum of K steps, meaning, under noise free condition, $\mathbf{r}^{k+1}$ becomes zero for some $k \leq K-1$. The K-sparse solution is then obtained as $\mathbf{\Phi}_{\Lambda^{k+1}}^\dagger\mathbf{y}$. Note that the gOMP algorithm boils down to the OMP algorithm [4] for $N = 1$.

III. PROPOSED IMPROVED BOUNDS

In gOMP algorithm, convergence in a maximum of K steps is established by ensuring that in each iteration, at least one of the N new atoms chosen belongs to $\mathbf{\Phi}_T$, i.e., it has an index belonging to the true support set T. Let $\beta^{k+1}, k = 0, 1, \cdots, K-1$ denote the largest (in magnitude) correlation between $\mathbf{r}^k$ and the atoms of $\mathbf{\Phi}_T$ at the k-th step of iteration, i.e., $\beta^{k+1} = max\{|\boldsymbol{\phi}_i^t\mathbf{r}^k| : i \in T, k = 0, 1, \cdots, K-1\}$. Also let the N largest (in magnitude) correlations between $\mathbf{r}^k$ and the atoms of $\mathbf{\Phi}$ not belonging to $\mathbf{\Phi}_T$ be given by $\alpha_i^{k+1}, i = 1, 2, \cdots, N$, arranged in descending order as $\alpha_1^{k+1} > \alpha_2^{k+1} \ldots > \alpha_N^{k+1}$. It is shown in [1] that

$$\alpha_N^{k+1} < \frac{\delta_{NK}}{1-\delta_{NK}} \frac{\|\mathbf{x}_{T-\Lambda^k}\|_2}{\sqrt{N}} \tag{2}$$

and

$$\beta^{k+1} > \frac{1-3\delta_{NK}}{1-\delta_{NK}} \frac{\|\mathbf{x}_{T-\Lambda^k}\|_2}{\sqrt{K-l}}, \tag{3}$$

where $l = |T \cap \Lambda^k|$. A sufficient condition to ensure convergence in a maximum of K steps is then obtained by setting the RHS of (3) greater than that of (2).

In this paper, we first retain the upper bound of α_N^{k+1} as given in (2), while the lower bound of β^{k+1} given in (3) is refined which eventually results in a lesser restrictive upper bound on δ_{NK} as shown in Theorem 1. Subsequently, we refine both the upper bound of α_N^{k+1} and the lower bound of β^{k+1}, which leads to an improved upper bound on δ_{NK+1} as shown in Theorem 2. The derivation uses the following Lemma, which is an extension of the Lemma 3.2 of [11].

Lemma 1: Given $\mathbf{u} \in \mathbb{R}^n$, $I_1, I_2 \subset Z$ where $I_2 = supp(\mathbf{u})$ and $I_1 \cap I_2 = \emptyset$,

$$\left(1 - \frac{\delta_{|I_1|+|I_2|}}{1-\delta_{|I_1|+|I_2|}}\right)\|\mathbf{u}\|_2^2 \leq \|\mathbf{A}_{I_1}\mathbf{u}\|_2^2 \leq (1+\delta_{|I_1|+|I_2|})\|\mathbf{u}\|_2^2.$$

Proof: Given in Appendix A. □

Additionally, we use certain properties of the RIP constant, given by Lemma 1 in [7], [8] which are reproduced below.

Lemma 2: For any two integers K_1, K_2 with $K_1 < K_2$, and, for $I, J \subset Z$ with $I \cap J = \emptyset$, $\mathbf{q} \in \mathbb{R}^{|I|}$, $\mathbf{p} \in \mathbb{R}^{|J|}$,

(a) $\delta_{K_1} \leq \delta_{K_2} \forall K_1 < K_2$ (monotonicity)
(b) $(1 - \delta_{|I|})\|\mathbf{q}\|_2 \leq \|\mathbf{\Phi}_I^t\mathbf{\Phi}_I\mathbf{q}\|_2 \leq (1 + \delta_{|I|})\|\mathbf{q}\|_2$
(c) $\langle\mathbf{\Phi}_I\mathbf{q}, \mathbf{\Phi}_J\mathbf{p}\rangle \leq \delta_{|I|+|J|}\|\mathbf{p}\|_2\|\mathbf{q}\|_2$ with equality holding if either of $\mathbf{p}$ and $\mathbf{q}$ is zero. Also, $\|\mathbf{\Phi}_I^t\mathbf{\Phi}_J\mathbf{p}\|_2 \leq \delta_{|I|+|J|}\|\mathbf{p}\|_2$ with equality holding if $\mathbf{p}$ is zero.

Theorem 1: The gOMP algorithm can recover $\mathbf{x}$ exactly when $\mathbf{\Phi}$ satisfies RIP of order NK with

$$\delta_{NK} < \frac{\sqrt{N}}{\sqrt{K}+2\sqrt{N}}.$$

Proof: First note that $\beta^{k+1} = \|\mathbf{\Phi}_T^t\mathbf{r}^k\|_\infty$, $k = 0, 1, \cdots, K-1$ and that $\mathbf{r}^k = \mathbf{P}_{\Lambda^k}^\perp\mathbf{y}$ is orthogonal to each column of $\mathbf{\Phi}_{\Lambda^k}$, which also means that $\mathbf{P}_{\Lambda^k}^\perp\mathbf{y} = \mathbf{P}_{\Lambda^k}^\perp\mathbf{\Phi}_T\mathbf{x}_T = \mathbf{P}_{\Lambda^k}^\perp(\mathbf{\Phi}_{T-\Lambda^k}\mathbf{x}_{T-\Lambda^k} + \mathbf{\Phi}_{T\cap\Lambda^k}\mathbf{x}_{T\cap\Lambda^k}) = \mathbf{P}_{\Lambda^k}^\perp\mathbf{\Phi}_{T-\Lambda^k}\mathbf{x}_{T-\Lambda^k}$. It is then possible to write,

$$\begin{aligned}
\beta^{k+1} &= \left\|\mathbf{\Phi}_T^t\mathbf{r}^k\right\|_\infty \\
&> \frac{1}{\sqrt{K-l}}\left\|\mathbf{\Phi}_{T-\Lambda^k}^t\mathbf{r}^k\right\|_2 \left(\text{as } |T-\Lambda^k| = K-l\right) \\
&= \frac{1}{\sqrt{K-l}}\left\|\mathbf{\Phi}_{T-\Lambda^k}^t\mathbf{P}_{\Lambda^k}^\perp\mathbf{y}\right\|_2 \\
&= \frac{1}{\sqrt{K-l}}\left\|\mathbf{\Phi}_{T-\Lambda^k}^t\left(\mathbf{P}_{\Lambda^k}^\perp\right)^t\mathbf{P}_{\Lambda^k}^\perp\mathbf{y}\right\|_2
\end{aligned}$$

$$= \frac{1}{\sqrt{K-l}} \left\| \left(\mathbf{P}_{\Lambda^k}^{\perp} \mathbf{\Phi}_{T-\Lambda^k}\right)^t \mathbf{P}_{\Lambda^k}^{\perp} \mathbf{\Phi}_{T-\Lambda^k} \mathbf{x}_{T-\Lambda^k} \right\|_2, \quad (4)$$

where we have used the fact that $\mathbf{P}_{\Lambda^k}^{\perp} = [\mathbf{P}_{\Lambda^k}^{\perp}]^t = [\mathbf{P}_{\Lambda^k}^{\perp}]^2$. Next we define a vector $\mathbf{x}'$, where $x_i' = x_i$ if $i \in T - \Lambda^k$ and $x_i' = 0$ otherwise. It is easy to see that $\mathbf{\Phi}\mathbf{x}' = \mathbf{\Phi}_{T-\Lambda^k}\mathbf{x}_{T-\Lambda^k}$ and thus,

$$\mathbf{A}_{\Lambda^k}\mathbf{x}' = \mathbf{P}_{\Lambda^k}^{\perp}\mathbf{\Phi}\mathbf{x}' = \mathbf{P}_{\Lambda^k}^{\perp}\mathbf{\Phi}_{T-\Lambda^k}\mathbf{x}_{T-\Lambda^k}. \quad (5)$$

[It is also easy to observe that $\mathbf{P}_{\Lambda^k}^{\perp}\mathbf{\Phi}\mathbf{x}' = \mathbf{P}_{\Lambda^k}^{\perp}\mathbf{\Phi}_T\mathbf{x}_T = \mathbf{r}^k$, since $\mathbf{P}_{\Lambda^k}^{\perp}\boldsymbol{\phi}_i = \mathbf{0}$ for $i \in \Lambda^k$.] We are now in a position to apply Lemma 1 on $\mathbf{A}_{\Lambda^k}\mathbf{x}'$, after taking $I_1 = \Lambda^k$, $I_2 = supp(\mathbf{x}') = T - \Lambda^k$ and noting that $I_1 \cap I_2 = \emptyset$, $|I_1| + |I_2| = Nk + K - l$, resulting in

$$\begin{aligned} \|\mathbf{A}_{\Lambda^k}\mathbf{x}'\|_2^2 &\geq \left(1 - \frac{\delta_{Nk+K-l}}{1-\delta_{Nk+K-l}}\right)\|\mathbf{x}'\|_2^2 \\ &\overset{L2a}{>} \left(1 - \frac{\delta_{NK}}{1-\delta_{NK}}\right)\|\mathbf{x}_{T-\Lambda^k}\|_2^2, \end{aligned} \quad (6)$$

where $Nk + K - l < NK$ follows from the fact that $k \leq l$ and $k < K$. Moreover,

$$\begin{aligned} \|\mathbf{A}_{\Lambda^k}\mathbf{x}'\|_2^2 &\overset{(5)}{=} \left\|\mathbf{P}_{\Lambda^k}^{\perp}\mathbf{\Phi}_{T-\Lambda^k}\mathbf{x}_{T-\Lambda^k}\right\|_2^2 \\ &= \left\langle \mathbf{P}_{\Lambda^k}^{\perp}\mathbf{\Phi}_{T-\Lambda^k}\mathbf{x}_{T-\Lambda^k}, \mathbf{P}_{\Lambda^k}^{\perp}\mathbf{\Phi}_{T-\Lambda^k}\mathbf{x}_{T-\Lambda^k} \right\rangle \\ &= \left\langle \left(\mathbf{P}_{\Lambda^k}^{\perp}\mathbf{\Phi}_{T-\Lambda^k}\right)^t P_{\Lambda^k}^{\perp}\mathbf{\Phi}_{T-\Lambda^k}\mathbf{x}_{T-\Lambda^k}, \mathbf{x}_{T-\Lambda^k} \right\rangle \\ &\leq \left\| \left(\mathbf{P}_{\Lambda^k}^{\perp}\mathbf{\Phi}_{T-\Lambda^k}\right)^t P_{\Lambda^k}^{\perp}\mathbf{\Phi}_{T-\Lambda^k}\mathbf{x}_{T-\Lambda^k} \right\|_2 \|\mathbf{x}_{T-\Lambda^k}\|_2. \end{aligned} \quad (7)$$

Combining (6) and (7) we get $\|(\mathbf{P}_{\Lambda^k}^{\perp}\mathbf{\Phi}_{T-\Lambda^k})^t P_{\Lambda^k}^{\perp}\mathbf{\Phi}_{T-\Lambda^k}\mathbf{x}_{T-\Lambda^k}\|_2 > (1 - \frac{\delta_{NK}}{1-\delta_{NK}})\|\mathbf{x}_{T-\Lambda^k}\|_2$.

From (4), it then follows that

$$\beta^{k+1} > \frac{1}{\sqrt{K-l}}\left(1 - \frac{\delta_{NK}}{1-\delta_{NK}}\right)\|\mathbf{x}_{T-\Lambda^k}\|_2. \quad (8)$$

Setting the RHS of (8) greater than the RHS of (2), the result follows trivially.[2] □

Theorem 2: The gOMP algorithm can recover $\mathbf{x}$ exactly when $\mathbf{\Phi}$ satisfies RIP of order $NK + 1$ with

$$\delta_{NK+1} < \frac{\sqrt{N}}{\sqrt{K} + \sqrt{N}}.$$

Proof: First, as seen earlier, $\mathbf{r}^k = \mathbf{P}_{\Lambda^k}^{\perp}\mathbf{y} = \mathbf{P}_{\Lambda^k}^{\perp}\mathbf{\Phi}_{T-\Lambda^k}\mathbf{x}_{T-\Lambda^k}$. We can then write,

$$\begin{aligned} \mathbf{r}^k &= \mathbf{\Phi}_{T-\Lambda^k}\mathbf{x}_{T-\Lambda^k} - \mathbf{P}_{\Lambda^k}\mathbf{\Phi}_{T-\Lambda^k}\mathbf{x}_{T-\Lambda^k} \\ &= \mathbf{\Phi}_{T-\Lambda^k}\mathbf{x}_{T-\Lambda^k} - \mathbf{\Phi}_{\Lambda^k}\mathbf{z}_{\Lambda^k} \\ &= \mathbf{\Phi}_{T\cup\Lambda^k}\mathbf{x}''_{T\cup\Lambda^k}, \end{aligned} \quad (9)$$

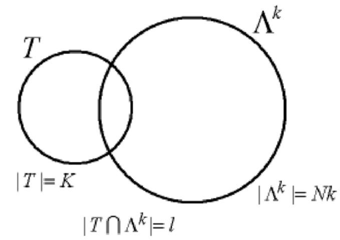

Fig. 1. Venn diagram for correct and estimated support set.

where we use the fact that $\mathbf{P}_{\Lambda^k}\mathbf{\Phi}_{T-\Lambda^k}\mathbf{x}_{T-\Lambda^k}$ belongs to the $span(\mathbf{\Phi}_{\Lambda^k})$ and thus can be written as $\mathbf{\Phi}_{\Lambda^k}\mathbf{z}_{\Lambda^k}$ for some $\mathbf{z}_{\Lambda^k} \in \mathbb{R}^{|\Lambda^k|} \equiv \mathbb{R}^{Nk}$. The vector $\mathbf{x}''_{T\cup\Lambda^k}$ is then given as

$$\mathbf{x}''_{T\cup\Lambda^k} = \begin{bmatrix} x_{T-\Lambda^k} \\ -z_{\Lambda^k} \end{bmatrix}. \quad (10)$$

More over we see that $\mathbf{r}^k = \mathbf{P}_{\Lambda^k}^{\perp}\mathbf{y}$ which implies that $\mathbf{r}^k \notin span(\mathbf{\Phi}_{\Lambda^k})$. Hence we can conclude that $\mathbf{r}^k$ belongs to such a subspace spanned by $(\mathbf{\Phi}_{T\cup\Lambda^k})$ which lies in null space of $span(\mathbf{\Phi}_{\Lambda^k})$.

Let W be the set of N incorrect indices corresponding to α_i^{k+1}'s for $i = 1, 2, \cdots, N$ (clearly, $W \subset \overline{T \cup \Lambda^k}$ and $|W| = N$). Then,

$$\begin{aligned} \alpha_N^{k+1} &= min\left(\left|\langle\Phi_i, \mathbf{r}^k\rangle\right| \big| i \in W\right) \\ &\leqslant \frac{\left\|\mathbf{\Phi}_W^t\mathbf{r}^k\right\|_2}{\sqrt{N}} \ (\text{as } |W| = N) \\ &\overset{(9)}{=} \frac{1}{\sqrt{N}}\left\|\mathbf{\Phi}_W^t\mathbf{\Phi}_{T\cup\Lambda^k}\mathbf{x}''_{T\cup\Lambda^k}\right\|_2 \\ &\overset{L2c}{\leqslant} \frac{1}{\sqrt{N}}\delta_{N+Nk+K-l}\|\mathbf{x}''_{T\cup\Lambda^k}\|_2 \\ &\overset{L2a}{<} \frac{1}{\sqrt{N}}\delta_{NK+1}\|\mathbf{x}''_{T\cup\Lambda^k}\|_2, \end{aligned} \quad (11)$$

where $N + Nk + K - l < NK + 1$ follows from the fact that $l \geq k$ and $k \leq K - 1$.

Similarly,

$$\begin{aligned} \beta^{k+1} &= \left\|\mathbf{\Phi}_T^t\mathbf{r}^k\right\|_\infty \geq \frac{1}{\sqrt{K}}\left\|\mathbf{\Phi}_T^t\mathbf{r}^k\right\|_2 \quad (\text{as } |T| = K) \\ &= \frac{1}{\sqrt{K}}\left\|[\mathbf{\Phi}_T \quad \mathbf{\Phi}_{\Lambda^k-T}]^t\mathbf{r}^k\right\|_2 \\ &= \frac{1}{\sqrt{K}}\left\|\mathbf{\Phi}_{T\cup\Lambda^k}^t\mathbf{\Phi}_{T\cup\Lambda^k}\mathbf{x}''_{T\cup\Lambda^k}\right\|_2 \\ &\overset{L2b}{>} \frac{1}{\sqrt{K}}(1 - \delta_{Nk+K-l})\|\mathbf{x}''_{T\cup\Lambda^k}\|_2 \\ &\overset{L2a}{>} \frac{1}{\sqrt{K}}(1 - \delta_{NK})\|\mathbf{x}''_{T\cup\Lambda^k}\|_2. \end{aligned} \quad (12)$$

Setting the RHS of (12) greater than that of (11), the result is obtained trivially. □

[2]Note that both [1] (Lemma 3.7) and the above procedure first derive a lower bound for β^{k+1} (β_1 in [1]) by applying a series of inequalities, with each inequality making the bound more restrictive than before. However, unlike [1] which relies on replacing $\mathbf{P}_{\Lambda^k}^{\perp}\mathbf{y}$ by $(\mathbf{y} - \mathbf{P}_{\Lambda^k}\mathbf{y})$ first and then on expanding $\mathbf{P}_{\Lambda^k}$ in its pseudo-inverse based form, the above procedure simply exploits the Hermitian as well as idempotent nature of $\mathbf{P}_{\Lambda^k}^{\perp}$ and also the Lemma 1, which leads to the deployment of much lesser number of inequalities than used in [1]. As a result, the lower bound of β^{k+1} derived above (i.e., (8)) turns out to be larger than as given in [1] which in turn leads to a lesser restrictive bound on δ_{NK}.

IV. Conclusions

In this letter, we have presented two new upper bounds for the RIP constant for use in the recently proposed gOMP algorithm [1], which guarantee exact reconstruction of a K-sparse signal in less than or equal to K number of iterations. The proposed bounds are obtained by suitable exploitation of the properties of the RIP constant and are considerably less restrictive than their counterparts existing in literature. Due to space limitations, the present treatment has, however, been confined to the noise free case only. It is nevertheless straightforward to extend this to the case of noisy observations by following arguments similar to [1].

Appendix A

Proof of Lemma 1

First, the relation is trivially satisfied with equality for $\mathbf{u}=\mathbf{0}$. We now assume that $\mathbf{u}\neq\mathbf{0}$. It is easy to see that for $\mathbf{u}\neq\mathbf{0}$, $\mathbf{\Phi u}$ can never be zero since $\mathbf{\Phi}$ satisfies RIP of order $|I_1|+|I_2|$ and thus of order $|I_2|$.

It is also easy to observe that $\|\mathbf{P}_{I_1}\mathbf{\Phi u}\|_2^2 = \langle \mathbf{P}_{I_1}\mathbf{\Phi u}, \mathbf{P}_{I_1}\mathbf{\Phi u}\rangle = \langle \mathbf{P}_{I_1}\mathbf{\Phi u}, \mathbf{\Phi u} - \mathbf{P}_{I_1}^{\perp}\mathbf{\Phi u}\rangle = \langle \mathbf{P}_{I_1}\mathbf{\Phi u}, \mathbf{\Phi u}\rangle$. Also, $\mathbf{P}_{I_1}\mathbf{\Phi u}\in span(\mathbf{\Phi}_{I_1})$, meaning $\mathbf{P}_{I_1}\mathbf{\Phi u}=\mathbf{\Phi z}$ for some $\mathbf{z}\in\mathbb{R}^n$ with $supp(\mathbf{z})\in I_1$. First consider the case where the columns of $\mathbf{\Phi}_{I_1}$ and those of $\mathbf{\Phi}_{I_2}$ are not orthogonal to each other, meaning $\mathbf{P}_{I_1}\mathbf{\Phi u}\neq\mathbf{0}$. We can then write,

$$\begin{aligned}\frac{\|\mathbf{P}_{I_1}\mathbf{\Phi u}\|_2}{\|\mathbf{\Phi u}\|_2} &= \frac{\langle \mathbf{P}_{I_1}\mathbf{\Phi u}, \mathbf{\Phi u}\rangle}{\|\mathbf{P}_{I_1}\mathbf{\Phi u}\|_2\|\mathbf{\Phi u}\|_2} = \frac{|\langle \mathbf{\Phi z}, \mathbf{\Phi u}\rangle|}{\|\mathbf{\Phi z}\|_2\|\mathbf{\Phi u}\|_2}\\ &\overset{D1}{\leq} \frac{|\langle \mathbf{\Phi z}, \mathbf{\Phi u}\rangle|}{\sqrt{1-\delta_{|I_1|}}\sqrt{1-\delta_{|I_2|}}\|\mathbf{u}\|_2\|\mathbf{z}\|_2} \overset{L2c}{\leq} \frac{\delta_{|I_1|+|I_2|}}{\sqrt{1-\delta_{|I_1|}}\sqrt{1-\delta_{|I_2|}}}\\ &\overset{L2a}{<} \frac{\delta_{|I_1|+|I_2|}}{1-\delta_{|I_1|+|I_2|}}.\end{aligned} \qquad \text{(A.1)}$$

This means, $\|\mathbf{A}_{I_1}\mathbf{u}\|_2^2 = \|\mathbf{\Phi u}\|_2^2 - \|\mathbf{P}_{I_1}\mathbf{\Phi u}\|_2^2 > (1-(\frac{\delta_{|I_1|+|I_2|}}{1-\delta_{|I_1|+|I_2|}})^2)\|\mathbf{\Phi u}\|_2^2 \overset{D1}{\geq} (1-(\frac{\delta_{|I_1|+|I_2|}}{1-\delta_{|I_1|+|I_2|}})^2)(1-\delta_{|I_2|})\|\mathbf{u}\|_2^2 \overset{L2a}{>} (1-\frac{\delta_{|I_1|+|I_2|}}{1-\delta_{|I_1|+|I_2|}}\|\mathbf{u}\|_2^2$ (if $\mathbf{P}_{I_1}\mathbf{\Phi u}=\mathbf{0}$, we directly have, $\|\mathbf{A}_{I_1}\mathbf{u}\|_2^2 = \|\mathbf{\Phi u}\|_2^2 \overset{D1}{\geq} (1-\delta_{|I_2|})\|\mathbf{u}\|_2^2 \overset{L2a}{>} (1-\frac{\delta_{|I_1|+|I_2|}}{1-\delta_{|I_1|+|I_2|}})\|\mathbf{u}\|_2^2$). Again, $\|\mathbf{A}_{I_1}\mathbf{u}\|_2^2 = \|\mathbf{\Phi u}\|_2^2 - \|\mathbf{P}_{I_1}\mathbf{\Phi u}\|_2^2 \leq \|\mathbf{\Phi u}\|_2^2 \overset{D1}{\leq} (1+\delta_{|I_2|})\|\mathbf{u}\|_2^2 \overset{L2a}{<} (1+\delta_{|I_1|+|I_2|})\|\mathbf{u}\|_2$.

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